Monday 2/24 Announcements & Reminders

- Homework 12, 13 & 14 due in class tomorrow
  - 19.1 j, 20.1 b, 20.4 will be graded for accuracy
  - Others for completion

- Quiz 3 tomorrow on homeworks 10 & 11
  - Derivatives
  - Power rule, product rule, quotient, chain etc…

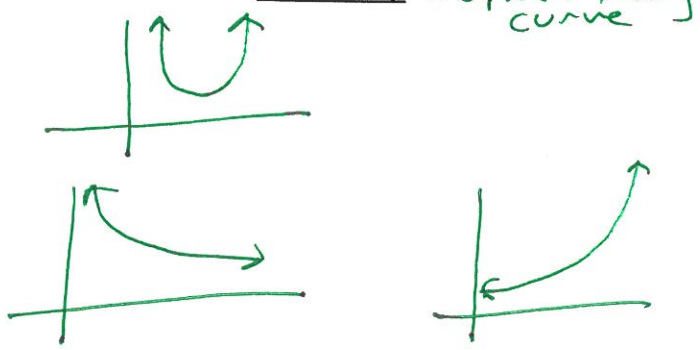
- I posted several derivative worksheets with solutions to blackboard for more practice/examples
  - You can find even more by googling “derivative examples”

- Project 2 still postponed a bit longer
  - Tomorrows class will be for quiz 3, homework questions, practice problems etc…

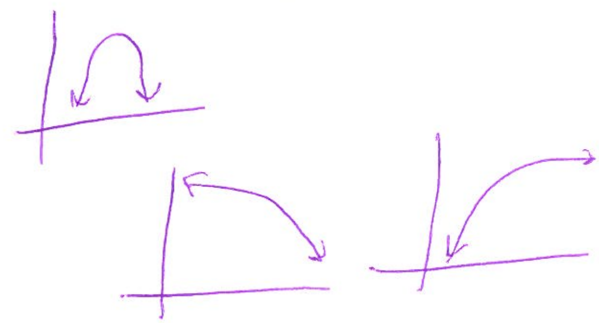
Concavity

The concept of concavity of a function is best grasped using pictures:

Concave Up - Upward Facing curve



Concave Down

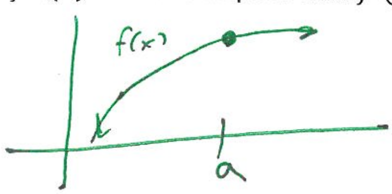


Another approach…

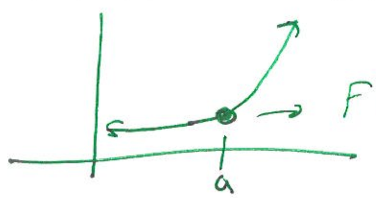
We know that $f'(x) =$ the rate of change of function f at point x

Then $(f'(x))' = f''(x) =$ the rate of change of the rate of change. That is, $f''(x)$ tell us how the rate of change is changing.

So if $f''(a) < 0$ this implies that $f'(x)$ is decreasing at $x = a$,which means that the graph is flattening out:



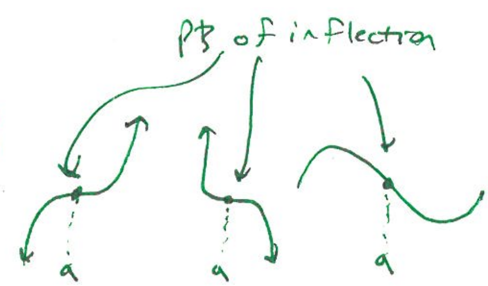
Similarly, if $f''(a) > 0$ then the graph is becoming more steep at $f(a)$:



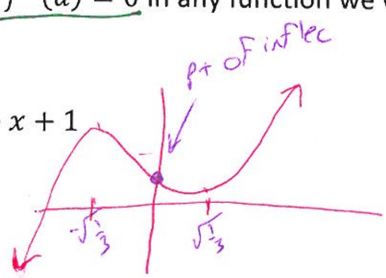
$f''(a) > 0$

Based on the above we state the following facts:

- If $f''(a) < 0$ then $f(x)$ is concave down at $x = a$
- If $f''(a) > 0$ then $f(x)$ is concave up at $x = a$
- If $f''(a) = 0$ then $f(x)$ is neither concave down nor up at $x = a$
  - Instead this is almost always a point of inflection (there are a few special cases where this is not true)
  - A Point of Inflection is a place where the graph switches concavity
  - If $f''(a) = 0$ in any function we will encounter, it will be a point of inflection



Ex: $g(x) = x^3 - x + 1$

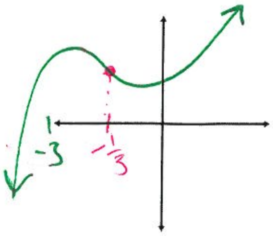


$g'(x) = 3x^2 - 1 = 0$ for $\boxed{x = \pm\sqrt{\frac{1}{3}}}$ → Can prove these are max/mins

$g''(x) = 6x = 0$

$\boxed{x = 0} \Rightarrow$ $\boxed{x = 0 \text{ is a pt of inflection}}$

Ex: Let $f(x) = x^3 + x^2 - x + 1$.
Determine the concavity of $f(x)$ at $x = -3$ and at $x = 0$. Where does $f(x)$ change concavity? $f''(x) = 0$?



$f'(x) = 3x^2 + 2x - 1$

$f''(x) = 6x + 2$

$f''(-3) = -18 + 2 = -16 < 0 \rightarrow$ concave down at $x = -3$

$f''(0) = 0 + 2 = 2 > 0 \rightarrow$ concave up at $x = 0$

$f''(x) = 6x + 2 = 0$ if $\boxed{x = -\frac{1}{3}}$

$\Rightarrow$ $\boxed{x = -\frac{1}{3} \text{ is a pt of inflection}}$

Recall that since max/mins are "flat" points on a graph, we solve $f'(x) = 0$ in order to find them.
So far we used $f'(x)$ to help classify critical points. We can also use $f''(x)$ to determine concavity at these values in order to classify them.

The Second Derivative Test (compare to 1st Deriv. Test)

Suppose $\boxed{f'(c) = 0} \rightarrow$ $f(c)$ is a flat point

- If $f''(c) > 0$ then $f(c)$ is a min
- If $f''(c) < 0$ then $f(c)$ is a max
- If $f''(c) = 0$ then $f(c)$ is a point of inflection

Examples

Let $h(t) = t^3 + t^2 - t + 1$. Find all critical point(s) and classify them using the second derivative test.
We have seen this problem previously so I won't spend time finding the critical points.

From last class, $h'(t) = 3t^2 + 2t - 1 = 0$ for $\boxed{t = \frac{1}{3},\ t = -1}$

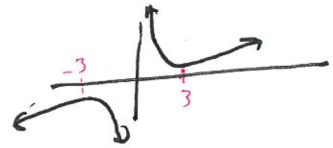
$h(t)$ & $h'(t)$ exist everywhere so these are only C.P.s

classify: $h''(t) = 6t + 2$

$h''(\frac{1}{3}) = 2 + 2 = 4 > 0 \Rightarrow$ $\boxed{t = \frac{1}{3} \text{ is a min}}$ concave up at $t = \frac{1}{3}$ & flat point

$h''(-1) = -6 + 2 = -4 < 0 \Rightarrow$ concave down at $t = -1$ & flat

$\Rightarrow$ $\boxed{t = -1 \text{ is a max}}$



$= x + 9x^{-1}$

Let $g(x) = x + \frac{9}{x}$. Find all critical point(s) and classify them using the second derivative test.

$g'(x) = 1 - 9x^{-2} = 0 \rightarrow 1 = \frac{9}{x^2} \xrightarrow{\text{cross multiply}} x^2 = 9 \Rightarrow \boxed{x = \pm\sqrt{9} = \pm 3}$

Also, $g(x)$ & $g'(x)$ DNE for $\boxed{x=0}$ $\Rightarrow$ crit. pts are $\boxed{x = 0, 3, -3}$

Classify: Since $x=0$ came from g/g' DNE, $\boxed{x=0 \text{ is a discontinuity}}$

$g'' = 18x^{-3} = \frac{18}{x^3}$

$g''(3) = \frac{18}{27} > 0 \longrightarrow$ concave up at $x=3$ $\Rightarrow \boxed{\text{min at } g(3)}$

$g''(-3) = \frac{18}{-27} < 0 \longrightarrow$ concave down at $x=-3$ $\Rightarrow \boxed{\text{max at } g(-3)}$

Let $g(x) = 1 + x^2 - \frac{x^6}{3}$. Find all critical point(s) and classify them using the second derivative test. Also, determine all values where $f(x)$ is concave up and concave down.

Finding the max/mins of a function on a closed interval

More often than not, Biological models have restricted domains and/or ranges.

For example:

- $M(w)$ which tells us the milligrams of medicine a person who weighs weight w requires

  Can't have negative milligrams nor negative weight => M in $[0,\infty)$, W in $(0,\infty)$

- Any time based model (ie, $f(t)$ where t is time)

  Need $t \geq 0$, ie, t is in $[0,\infty)$

- Probability function $P(x)$ (from 151)

  $P(x)$ needs to be in $[0,1]$

Procedure for finding the max/mins of function $f(x)$ on a closed interval $[a, b]$:

1) Find all ~~max/mins~~ crit. pts of $f(x)$ as discussed previously (take deriv. solve $f'=0$)
2) Eliminate any that are not found in the interval $[a, b]$
3) ~~Also check the end points $f(a)$ and $f(b)$ to see if they happen to be higher/lower than the max/min you found in 1) and 2)~~ plug all c.p.s and a, b in $f(x)$ to see largest/smallest

Ex: Let $f(x) = x^4 - 2x^2$. Find the maximum and minimum of $f(x)$ on the interval $[-1, 2]$.

① $f'(x) = 4x^3 - 4x = 0$ $\quad x(4x^2-4)=0$

$\Rightarrow \boxed{x=0}$ or $4x^2-4=0$

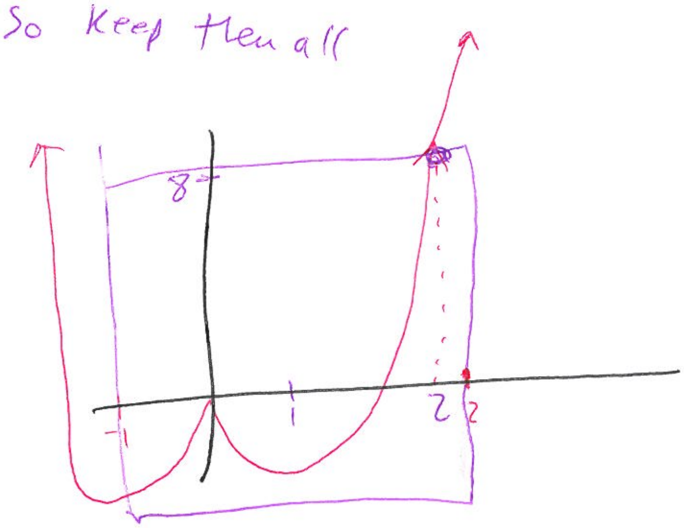
$\Rightarrow \boxed{x=\pm 1}$

② all c.p.s are in $[-1,2]$ so keep them all

③ $f(0)=0$

$\left.\begin{array}{l} f(1)=1-2=-1 \\ f(-1)=1-2=-1 \end{array}\right\}$ min at $x=-1$

end points $f(2)=16-8=8 \}$ max at $x=2$



Remarks for homework

① We find critical points by
- Seeing where $f(x)$ & $f'(x)$ DNE
- solving $f'(x) = 0$.

ⓐ If $f'(x)$ is a polynomial, solve $f'(x)=0$ by factoring, quadratic formula, etc...

ⓑ If $f'(x) = \frac{g(x)}{h(x)}$ (ie, $f'(x)$ is a fraction), Solve $f'(x)=0$ by setting $g(x)=0$. (Set the numerator $= 0$)

ⓒ If $f'(x) = ax^n + \frac{b}{x^n}$ Then Solve $f'(x)=0$ using method from todays notes

② To classify a critical point found where $f'(x)=0$ we use:

ⓐ The First Derivative Test ⟶ Learned Last Wednesday (2/19)

and/or

ⓑ The Second Derivative Test ⟶ Learned today (Mon 2/24)

③ Critical Points found where $f(x)$ and/or $f'(x)$ DNE are always discontinuities.

Note: Use graphing as a visual aid to check your work. However, resuts need to be supported using calculus techniques.

Wednesday 2/26 Announcements & Reminders

- Quiz 4 on homework 12, 13, 14 & 15 next Tuesday 3/4
  - Finding critical points
  - Classifying them using 1st and 2nd derivative tests

- Homework 15 due Tuesday 3/4
  - Possibly only one question on quiz from this homework

- We will have time during the next several classes to answer questions, go over homework problems, see more examples etc
  - Submit questions/topics via email
  - I'll also give out a survey soon

- I have an appointment tomorrow to get a receiver for polling. Bring your clicker to class next week!
  - Math 152 is gonna be AWESOMEEE (hopefully)

Remarks for homework

① We find $\boxed{\text{critical points}}$ by • Seeing where $f(x)$ & $f'(x)$ DNE
• Solving $f'(x) = 0$.

ⓐ If $f'(x)$ is a polynomial, solve $f'(x)=0$ by factoring, quadratic formula, etc...

ⓑ If $\boxed{f'(x) = \frac{g(x)}{h(x)}}$ (ie, $f'(x)$ is a fraction), solve $f'(x)=0$ by setting $\boxed{g(x)=0}$. (Set the numerator $=0$)

ⓒ If $\boxed{f'(x) = ax^n + \frac{b}{x^n}}$ Then solve $f'(x)=0$ using method from todays notes
Monday

② To classify a critical point found where $\boxed{f'(x)=0}$ we use:

ⓐ The First Derivative Test $\longrightarrow$ Learned Last Wednesday (2/19)

and/or

ⓑ The Second Derivative Test $\longrightarrow$ Learned today (Mon 2/24)

③ Critical Points found where $f(x)$ and/or $f'(x)$ DNE are always discontinuities.

$\boxed{\text{Note}}$: Use graphing as a visual aid to check your work. However, results need to be supported using calculus techniques.

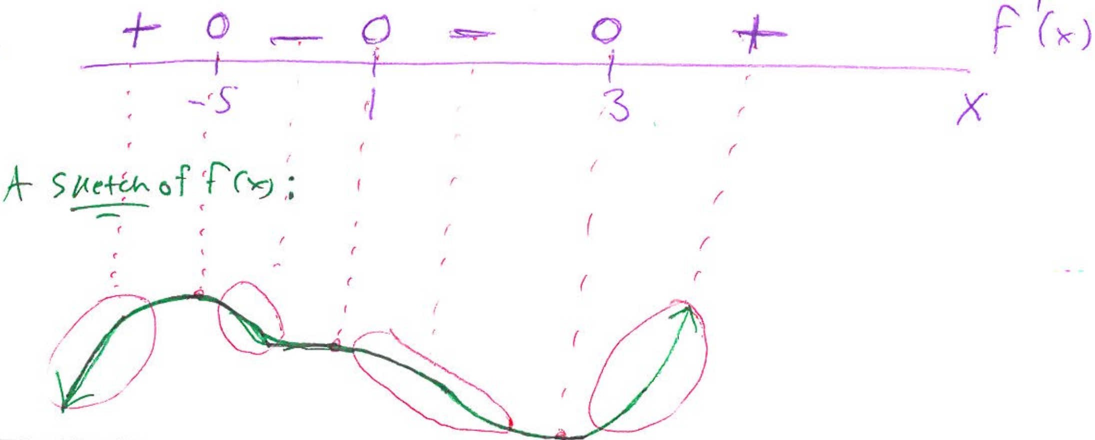
Sketching $f(x)$ given information about $f'(x)$

Since $f'(x)$ tells us where $f(x)$ is increasing/decreasing and where the maximums/minimums are, we can use this information to sketch the general shape of $f(x)$.

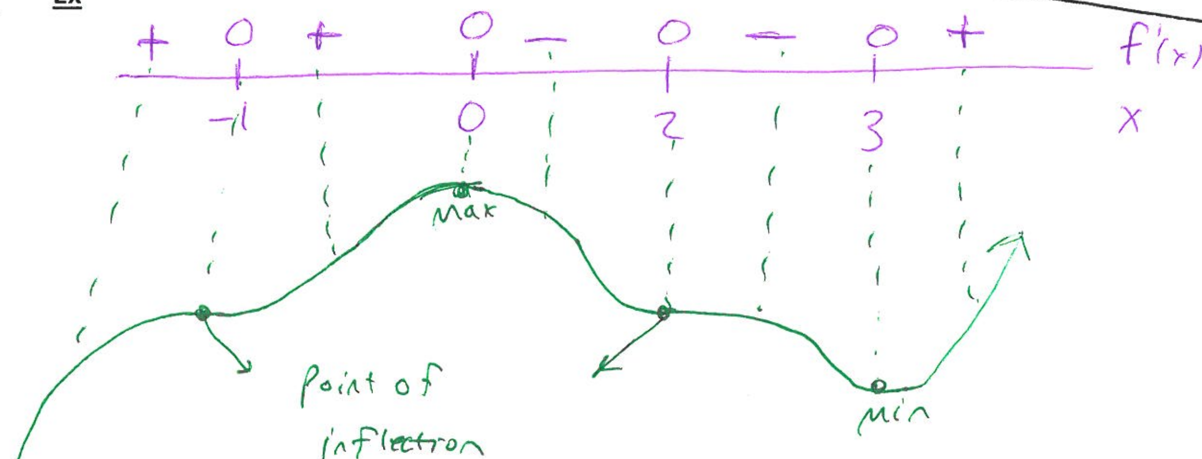
Ex: Suppose we used the 1st derivative test and obtained the following sign chart:

x	$f'(x)$
-6	>0
-5	0
-4	<0
1	0
2	<0
3	0
4	>0

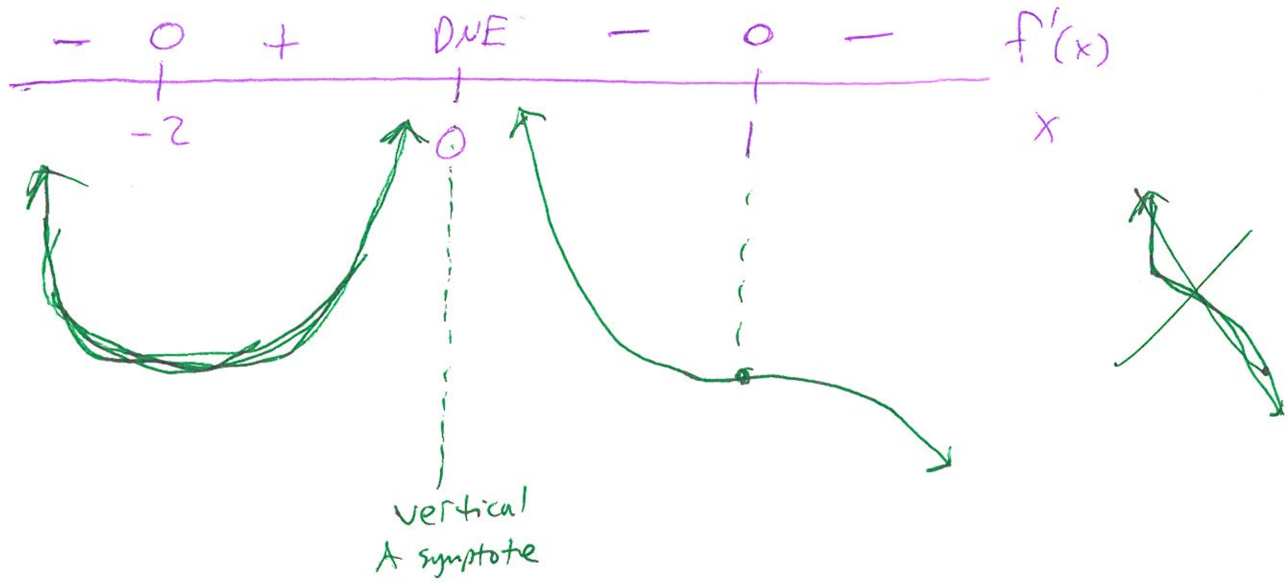
or



Ex



Ex (assume the discontinuities are infinite discontinuities)



$\square$ Something we are Maximizing/Minimizing

$\square$ Constraint on above objective

Optimization Problems

Getting the most out of a situation is something everyone strives for. This is especially the case when considering complex systems that contain objectives and constraints whose relationship with each other have unclear tradeoffs and implications. Such systems arise in countless applications in sciences, business and everyday life.

- What levels of light and/or nutrients yield the best crops?
- What is the best way to get our product to the distributors? "Logistic"
- Given a limited quantity of supplies, how can we make the most effective or most cost efficient or largest product? ex 100 m² of boxing material to package our product
- What dosage/frequency of a drug will effectively fight the disease without hurting the patient or resulting in the patient becoming broke?
- What is the least amount of effort I can put into math 152 while still getting the grade I want?
- How can we employ our resources to limit the population of an invasive pest while efficiently managing our funds?

Since a major application of calculus is finding the maximums/minimums of functions, it provides us a tool in which to answer some of these types of questions so that we can achieve the optimal result.

All optimization problems are different (which is ~~be~~ fun and exciting!)

What follows are general guidelines to solving optimization problems:

- Some steps don't always apply
- Some steps are sometimes done for you

1) Define the key variables in the problem
   - There can be many variables in real world applications
   - We will generally only see 2 in each problem
2) Draw a picture relating the variables (if applicable)
3) Derive the objective function $Q(x)$
   - This is the function that we are trying to maximize/minimize
   - This function should be written in terms of the variables found in 1)
4) Determine if there exist any constraints on the variables
5) Write $Q(x)$ in terms of only 1 variable
   - Usually requires solving for one variable and substituting it into the other
6) Use calculus to find the max/min of $Q(x)$
7) Make sure your answer is reasonable
   - Help catch mistakes and/or make improvements

Examples on this page have most steps done for you.

Optimization Examples:

Suppose the function $Y(N) = \frac{N}{1+N^2}$ models the yield Y of a crop given the nitrogen level N.

What Nitrogen level yields the most crops? → Find N^* such that it maximizes Yield. ie, find the max of $Y(N)$

Constraints: need Y in $[0,\infty)$ and N in $[0,\infty]$

$$Y' = \frac{1\cdot(1+N^2) - N(2N)}{(1+N^2)^2} = \frac{1+N^2-2N^2}{(1+N^2)^2} = \frac{1-N^2}{(1+N^2)^2}$$

Quotient rule

$Y' = 0$ if $1-N^2 = 0 \Rightarrow N^2 = 1 \Rightarrow \boxed{N = \pm 1}$ → Potential max/mins

$N=-1$ is outside our constraint for N so don't consider it

Verify that $N=1$ is a max

N	$Y'(N)$
0	$1 > 0$
1	0
2	$-3/25 < 0$

$\Rightarrow$ $\boxed{N=1 \text{ is a max}}$

$\boxed{N=1 \text{ yields the most crops}}$

whats the max yield?

$\Rightarrow Y(1) = \frac{1}{1+1^2} = \frac{1}{2}$

Suppose $C(t)$ tells use the concentration of a drug in the blood stream t hours after injecting it as given by

$$C(t) = \frac{6t}{2t^2+1}$$

What is the highest concentration the drug will achieve in the blood steam?

Monday 3/3 Announcements & Reminders

- I am starting to use interactive polling in our classes.
  - I will keep track of attendance and participation
  - Respon~~s~~ding will help you stay engaged in the class and help me understand how to pace the material.

Channel 58

Upcoming Dates

- This Tuesday 3/4
  - Homework 15 due
  - Quiz 4 on homework 12, 13, 14 & 15
    - There focus on finding critical points
    - classifying critical points
      - using 1^{st} and 2^{nd} derivative tests

- Test 2 is next Tuesday 3/11
  - Test and solutions from last semester posted to blackboard

Common Mistakes

- $\boxed{\dfrac{a+b}{c}} = \dfrac{a}{c} + \dfrac{b}{c}$ ex) $\dfrac{x^2+1}{x^2} = \dfrac{x^2}{x^2} + \dfrac{1}{x^2} = \boxed{1+\dfrac{1}{x^2}}$

  Not: $\cancel{\dfrac{x^2+1}{x^2} = \dfrac{1+1}{1}}$

- $\dfrac{a}{c+b}$ ex) $\dfrac{x^2}{x^2+1}$ → can't break up fraction

- $(x^2+1)^3 \neq x^3+1^3$

  $(x^2+1)^3 = (x^2+1)(x^2+1)(x^2+1)$

- $x^2 = 4 \Rightarrow x = \pm\sqrt{4} = \boxed{\pm 2}$

  $x^3 = 8 \Rightarrow x = \sqrt[3]{8} = 2$ $\cancel{\pm}$

- $\ln$, $\sin$, $\cos$ alone doesn't make sense

ie, $\ln(\cdot)$, $\sin(\cdot)$, $\cos(\cdot)$ need an argument inside them

$\boxed{ex}$ $\ln(x)$, $\sin(x)$, $\cos(x)$

$\ln(x^2+1)$, $\sin(e^x)$, $\cos(\sqrt{x}+\pi)$

- $e^{(\cdot)}$ does not need an argument since $e \approx 2.71$

So $e^x \approx 2.71^x$

or $e^1 \approx 2.71$

ex $f(x) = \sin(x^2) \rightarrow$ need to use ~~(sine rule)~~ chain rule for f'

A

$f(x) = (x^3+1)^2$

Find critical points

$f'(x) = 2(x^3+1)^1\left(\frac{d}{dx}x^3+1\right) = 2(x^3+1)(3x^2)$

$f'(x)=0$ if $2x^3+2=0 \iff x^3=-1 \iff \boxed{x=-1}$
or $3x^2=0 \iff x^2=0 \iff \boxed{x=0}$
$\Rightarrow$ Places in $f(x)$ with zero rate of change, ie, "flat" spots on the graph

1st Derivative Test

x	$f'(x)$
-2	<0
-1	$=0$
$-.5$	>0

$\Rightarrow$ A min at -1

$-\ 0\ +$ $f'(x)$, at $x=-1$

x	$f'(x)$
$-.5$	>0
0	0
1	>0

$\Rightarrow$ Inflection Point at $x=0$

$+\ 0\ +$ $f'(x)$, at $x=0$

2nd Deriv Test:

$f''(x) = 30x^4+12x$

$f''(-1) = 18 > 0 \Rightarrow$ Min at -1

$f''(0) = 0 \Rightarrow$ Inflection point at $x=0$

B $h(t) = \dfrac{e^t}{t^2+2t}$

Critical points

$h'(t) = \dfrac{e^t(t^2+2t)-e^t(2t+2)}{(t^2+2t)^2} = \dfrac{e^t(t^2-2)}{(t^2+2t)^2}$

$h'(t)=0$ if $e^t(t^2+2t)-e^t(2t+2)=0$
$e^t(t^2+2t-2t-2)=0$
$e^t(t^2-2)=0 \iff$

$e^t=0$ ↳ not possible $\qquad t^2-2=0 \iff \boxed{t=\pm\sqrt{2}}$

also, $(t^2+2t)^2=0 \iff (t(t+2))^2=0 \iff \boxed{t=0,\ t=-2}$

Since $f(0)$ and $f(-2)$ DNE, they are points of discontinuity

1 Derivative Test

x	$f'(x)$
-1.5	>0
$-\sqrt{2}$	0
-1	<0

$\Rightarrow$ a max at $-\sqrt{2}$

$+\ \ -$ $f'(x)$, at $x=-\sqrt{2}$

x	$f'(x)$
1	<0
$\sqrt{2}$	0
3	>0

$\Rightarrow$ a min at $x=\sqrt{2}$

$-\ \ +$ $f'(x)$, at $x=\sqrt{2}$

2nd Derivative Test

$f''(x) = \dfrac{e^t(t^4-2t^2+4t+8)}{t^3(t+2)^3}$

$f''(-\sqrt{2}) < 0 \Rightarrow$ max at $x=-\sqrt{2}$

$f''(\sqrt{2}) > 0 \Rightarrow$ min at $\sqrt{2}$

Since we tested all c.p.s and $f''(x) \neq 0$ anywhere, there are no inflection points

More Test Stuff

Overview of Derivatives

a) Increasing/Decreasing found using 1^{st} Derivative $(f'(x))$

b) Concavity found using 2^{nd} Derivative $(f''(x))$

Critical Points

c) Max/mins found solving $f'(x)=0$
& classified using either the 1^{st} or 2^{nd} derivative test

d) Discontinuities found where $f'(x)/f(x)$ DNE

e) Inflection point (where the function changes concavity)
Found where $f''(x)=0$ and/or $\underset{a}{+ \quad 0 \quad +}\ f'(x)$ / $\underset{a}{- \quad 0 \quad -}\ f'(x)$

Chain Rule:

either a) $e^{g(x)}$, $\sin(g(x))$, $\cos(g(x))$ or $\ln(g(x)) \to$ See Formula Sheet

$\boxed{ex}$ $e^{\frac{1}{x}}$. $\frac{d}{dx} e^{\frac{1}{x}} = \left(\frac{d}{dx}\frac{1}{x}\right)e^{1/x} = -\frac{1}{x^2}e^{1/x}$

$\boxed{ex}$ $\frac{d}{dx}\cos(2x^2+4\sqrt{x}) = \left(\frac{d}{dx}2x^2+4\sqrt{x}\right)\left(-\sin(2x^2+4\sqrt{x})\right)$

$= \left(4x+2\frac{1}{\sqrt{x}}\right)\left(-\sin(2x^2+4\sqrt{x})\right)$

or b) $[g(x)]^{\wedge}$

$\boxed{ex}$ $f(x) = \left(x^2+\sqrt{x}+\frac{1}{x^2}\right)^3$

$\Rightarrow f'(x) = 3\left(x^2+x^{1/2}+x^{-2}\right)^2\left(\frac{d}{dx}x^2+x^{1/2}+x^{-2}\right)$

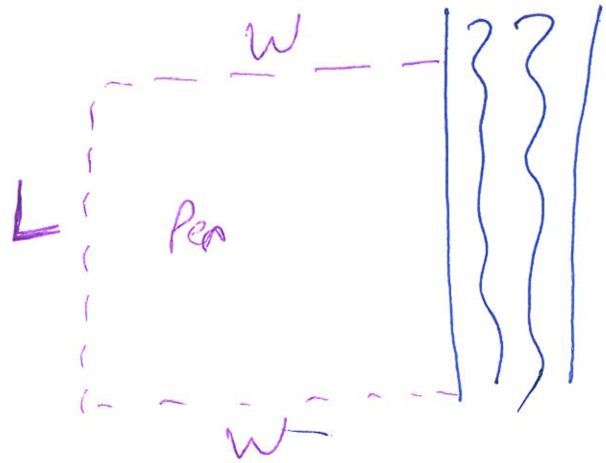
$= 3\left(x^2+\sqrt{x}+\frac{1}{x^2}\right)^2\left(2x+\frac{1}{2\sqrt{x}}-\frac{2}{x^3}\right)$

More Optimization

A farmer has a grazing area and has 3000 feet of fencing to make a rectangular pen. The grazing area is next to a river so the farmer only needs to construct 3 walls. What height and width of the pen would maximize the area of the pen?

L = length of side next to river

W = length of other 2 sides



Objective function

want to max area

where $\boxed{\text{Area} = L \cdot W}$ → maximize this

Constraints

- need $L > 0$, $W > 0$
- want $L + W + W = 3000$ or $\boxed{L + 2W = 3000}$

Need $A = L \cdot W$ in terms of 1 variable

So let $L = \underbrace{3000 - 2W}$ → sub into $A = L \cdot W$

So $A = L \cdot W = (3000 - 2W)W = 3000W - 2W^2$

Maximize $A = 3000W - 2W^2$. $A' = 3000 - 4W = 0$

$3000 = 4W \Rightarrow \boxed{W = 750 \text{ ft}}$

verify W is a max: $A'' = -4 \Rightarrow A''(750) = -4$

$\Rightarrow W = 750$ is a max by 2nd deriv test

Since $W = 750$ and $2W + L = 3000 \Rightarrow 2(750) + L = 3000$

$\Rightarrow \boxed{L = 1500 \text{ ft}}$

So $L = 1500$, $W = 750$ yield max area of $\boxed{A = 1500 \cdot 750 = 1{,}125{,}000 \text{ ft}^2}$

More Optimization

The average individual daily milk consumption for Charolais, Angus and Hereford calves is approximated by the function

$$M(t) = 6.281t^{0.242}e^{-0.025t} \quad , \qquad \boxed{1 \le t \le 26}$$

Where $M(t)$ is the milk consumption (in kg) and t is the age of the calf (in weeks).
1. Find the age of a calf at which maximum daily consumption occurs.
2. How much milk is consumed on this day?
3. Do you expect this value to be exactly the same for all calves?

① Find $M'(t)$ and solve $M' = 0$

$$M'(t) = \left(\frac{d}{dt}\, 6.28\, t^{.242}\right) e^{-.025t} + \left(6.28\, t^{.242}\right)\cdot\left(\frac{d}{dt} e^{-.025t}\right)$$

$$= (.242)(6.28)\left(t^{.242-1}\right) e^{-.025t} + \left(6.28\, t^{.242}\right)\left(-.025 e^{-.025t}\right)$$

$$= e^{-.025t}\left[1.52\, t^{-.758} - .157\, t^{.242}\right] = 0$$

$$\underbrace{e^{-.025t} = 0}_{\neq 0 \text{ for any } t} \quad \text{or} \quad 1.52\, t^{-.758} - .157\, t^{.242} = 0$$

$$\cancel{t^{.758}} \cdot \frac{1.52}{\cancel{t^{.758}}} = .157\, t^{.242} \cdot t^{.758}$$

$$1.52 = .157\, t^{.242+.758} \Rightarrow \boxed{t = \frac{1.52}{.157} \doteq 9.68}$$

Verify $t = 9.68$ is a max

x	f'
8	>0
10	<0

$\Rightarrow M(9.68)$ is a max

$\boxed{\text{a 9.68 week old calf drinks most milk}}$

② $M(9.68) = 8.5$ kg of milk is most consumed

__Wed 3/5 Announcements & Reminders__

- I have quiz 4 to pass back. Come get it before class because we will go over it during class

- Test 2 is next Tuesday 3/11

- Solutions to all homework problems will become available this afternoon at 5 p.m.

    - Homework 15 will be graded soon
    - Homework 16 & 17 will not be collected for a grade. Optimization problem(s) on the test will be similar to a homework problem or class problem.

- Office hours Monday night (3/10)? 11-3 email me times you want to come

- Submit questions for Monday via email

- We will go over project 2 next Wednesday 3/12

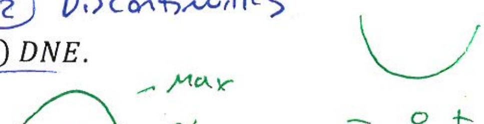
Math 152 Test 2 Topics

Topics: example test 2 on 6.6

1) Finding derivatives of functions using derivative rules
   a. I will provide the $\boxed{\text{quotient rule and chain rule ONLY.}}$ You need to know the rest
   b. Be prepared to take multiple derivatives

2) Finding the equation of a tangent line to $f(x)$ at point $x = a$
   a. Follow the 3 steps specified in the notes

3) Finding critical points ① Max/min I.P. ② Discontinuities
   a. Occur where $f'(x) = 0$ and where $f'(x)\ DNE$.



4) Classifying Critical points- Two Methods
   a. First Derivative Test
      i. Test points on either side of each critical point to determine where $f(x)$ is increasing/decreasing to see if we have a max or min.
   b. Second Derivative Test
      i. Test the critical point in the second derivative to determine concavity at the critical point to determine if we have a max or min.

- Max: $+ \; 0 \; -$ f' (at a, x)
- min: $- \; 0 \; +$ f' (at a, x)

classify

$f''(c.p.) > 0 \Rightarrow$ min $\qquad f''(c.p.) < \Rightarrow$ max

5) Concavity
   a. Determined using second derivative
      i. $f''(a) < 0 \Rightarrow f(x)$ concave down at $f(a)$
      ii. $f''(a) > 0 \Rightarrow f(x)$ concave up at $f(a)$
      iii. $f''(a) = 0 \Rightarrow$ ~~possible~~ point of inflection at $f(a)$ or

$+ \; 0 \; +$ f' (at a, x) or $- \; 0 \; -$ f' (at a, x)

   b. To determine all values where $f(x)$ is concave up/down set $f''(x) < 0$ and $f''(x) > 0$ and solve for resulting values
      down $\qquad$ up

6) Knowing what $f'(x)$and $f''(x)$ tell us about $f(x)$
   a. $f'(x)$
      i. Tells us the instantaneous rate of change of $f(x)$ aka the slope of the tangent line at point x. aka "steepness" of the graph at point x
      ii. Has units $\frac{units\ of\ f(x)}{units\ of\ x}$

b. $f''(x)$

   i. Tells us how the first derivative is changing. That is, tells us the rate of change of the rate of change. This can be used to determine concavity

   ii. Has units $\dfrac{\frac{units\ of\ f(x)}{units\ of\ x}}{units\ of\ x} = \boxed{\dfrac{units\ of\ f(x)}{(units\ of\ x)^2}}$

7) Finding the max/min of $f(x)$ on a closed interval.
   a. Find all critical points
   b. Test critical points found in the interval **&** end points of the interval in $f(x)$
   c. Choose the largest/smallest as your max/min

8) Sketching $f(x)$ given info about $f'(x)$

9) Maximization/Minimization Problems

Good Practice Problems:

- All examples from class
- All examples in each chapter (these are good because they have detailed answers)
- All homework problems (graded and non-graded)
- Derivative worksheets on blackboard (answer keys included)

Ch 18:

- 18.3, 18.4, 18.5

Ch 19

- All problems from 19.1 through 19.6
- 19.12

Ch 20

- 20.1
- 20.2
- 20.3
- 20.4
- 20.5
- 20.7
- 20.14
- 20.16
- 20.18 and 20.19

More Test Stuff

Overview of Derivatives

ⓐ Increasing / Decreasing found using 1^{st} Derivative $(f'(x))$

ⓑ Concavity found using 2^{nd} Derivative $(f''(x))$

Critical Points

ⓒ Max/mins found solving $f'(x)=0$
& classified using either the 1^{st} or 2^{nd} derivative test

ⓓ Discontinuities found where $f'(x)/f(x)$ DNE

ⓔ Inflection point (where the function changes concavity)
found where $f''(x)=0$ and/or $\frac{+ \quad 0 \quad +}{a} f'(x)$ / $\frac{- \quad 0 \quad -}{a} f'(x)$

Chain Rule:

either ⓐ $e^{g(x)}$, $\sin(g(x))$, $\cos(g(x))$ or $\ln(g(x))$ → See Formula Sheet

$\boxed{ex}$ $e^{\frac{1}{x}}$. $\frac{d}{dx} e^{\frac{1}{x}} = \left(\frac{d}{dx}\frac{1}{x}\right)e^{1/x} = -\frac{1}{x^2}e^{1/x}$

$\boxed{ex}$ $\frac{d}{dx}\cos(2x^2+4\sqrt{x}) = \left(\frac{d}{dx}2x^2+4\sqrt{x}\right)\left(-\sin(2x^2+4\sqrt{x})\right)$

$= \left(4x+2\frac{1}{\sqrt{x}}\right)\left(-\sin(2x^2+4\sqrt{x})\right)$

or ⓑ $[g(x)]^{n}$

$\boxed{ex}$ $f(x) = \left(x^2+\sqrt{x}+\frac{1}{x^2}\right)^3$

$\Rightarrow f'(x) = 3\left(x^2+x^{1/2}+x^{-2}\right)^2\left(\frac{d}{dx}x^2+x^{1/2}+x^{-2}\right)$

$= 3\left(x^2+\sqrt{x}+\frac{1}{x^2}\right)^2\left(2x+\frac{1}{2\sqrt{x}}-\frac{2}{x^3}\right)$

Examples on this page have most steps done for you.

Optimization Examples:

Suppose the function $Y(N) = \frac{N}{1+N^2}$ models the yield Y of a crop given the nitrogen level N.

What Nitrogen level yields the most crops? → Find N^* such that it maximizes Yield. ie, Find the max of $Y(N)$

Constraints: need Y in $[0,\infty)$ and N in $[0,\infty)$

$$Y' = \frac{1\cdot(1+N^2) - N(2N)}{(1+N^2)^2} = \frac{1+N^2-2N^2}{(1+N^2)^2} = \frac{1-N^2}{(1+N^2)^2}$$

Quotient rule

$Y'=0$ if $1-N^2=0 \Rightarrow N^2=1 \Rightarrow \boxed{N=\pm 1}$ → potential max/mins

$N=-1$ is outside our constraint for N so don't consider it

Verify that $N=1$ is a max

N	$Y'(N)$
0	$1>0$
1	0
2	$-3/25<0$

$\Rightarrow$ $\boxed{N=1 \text{ is a max}}$

$\boxed{N=1 \text{ yields the most crops}}$

whats the max yield?

$\Rightarrow Y(1) = \frac{1}{1+1^2} = \frac{1}{2}$

Suppose $C(t)$ tells use the concentration of a drug in the blood stream t hours after injecting it as given by

$$C(t) = \frac{6t}{2t^2+1}$$

find max of $C(t)$

What is the highest concentration the drug will achieve in the blood steam?

$$C'(t) = \frac{6(2t^2+1) - 6t(4t)}{(2t^2+1)^2} = \frac{-12t^2+6}{(2t^2+1)^2}$$

$C'(t)=0$ for $-12t^2+6=0 \Rightarrow \boxed{t=\pm\sqrt{\tfrac{1}{2}}}$

Classify

t	C'
-1	$-2/3<0$
0	$6>0$
1	$-\frac{2}{3}<0$

C': $-$ 0 $+$ 0 $-$

t: -1, $\boxed{-\sqrt{\tfrac{1}{2}}}$, 0, $\boxed{\sqrt{\tfrac{1}{2}}}$, 1

$\Rightarrow$ $\boxed{\text{max at } t=\sqrt{\tfrac{1}{2}}}$

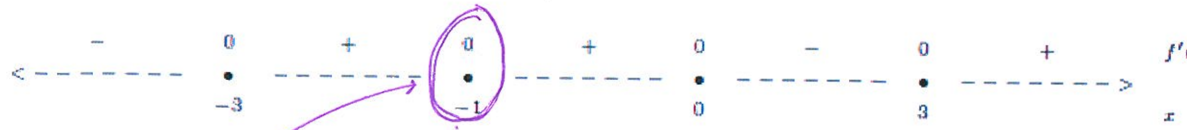
Max concentration occurs at $t=\sqrt{\tfrac{1}{2}}$

has a value of $C(\sqrt{\tfrac{1}{2}}) = \frac{6\sqrt{\frac{1}{2}}}{2\cdot\frac{1}{2}+1} = \boxed{2.12}$ ← answer

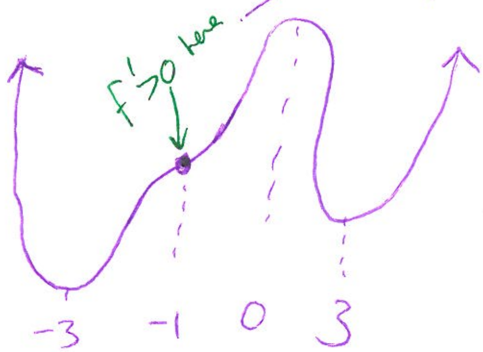
Math 152 Quiz 4

Name Solutions

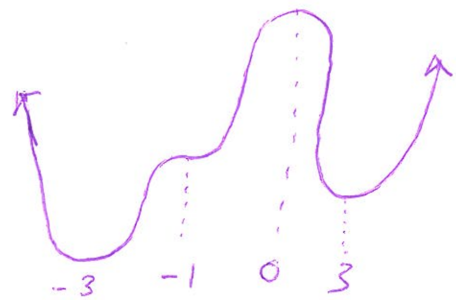
1) Given the following sign chart for $f'(x)$, choose which option is the best representation of the general shape of $f(x)$. (2pts)



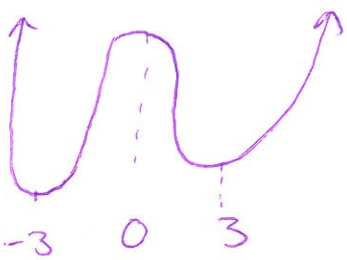
a)



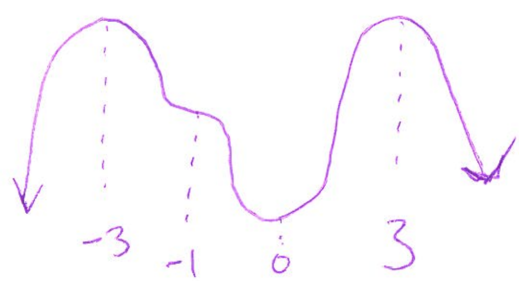
c)



b)

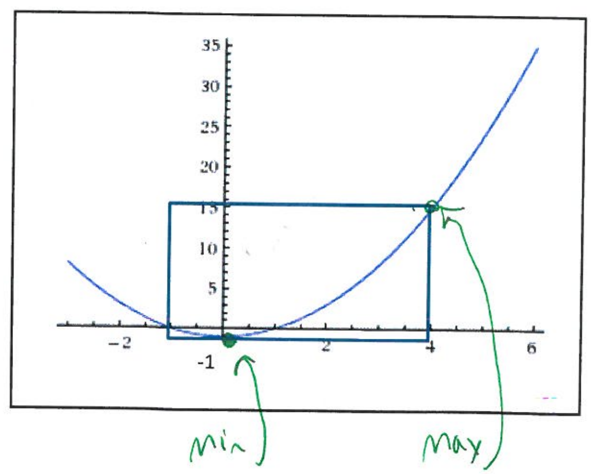


d)



2) Find the absolute maximum and minimum of $f(x) = x^2 - 1$ on the interval $[-1,4]$.

The graph of $f(x)$ is show on the right with the box highlighting the interval $[-1,4]$. (2pts)



a) min of -1, max of 15

b) min of -1, max of 35

c) min of -1, no max

d) min of 15, max of -1

e) min of -1, max of 0

(3pts)

3) Let $f(x) = e^{x^2}$. Find $f'(x)$. → use exponential rule

$$f' = \left(\frac{d}{dx}x^2\right) e^{x^2} = 2x e^{x^2}$$

(3pts)

4) Let $f(x) = x + \frac{1}{x}$. Find all critical points of $f(x)$. → ① Solve $f' = 0$ ② See where f/f' DNE

① $f = x + x^{-1} \Rightarrow f' = 1 - x^{-2} = 0$

$\Rightarrow 1 = \frac{1}{x^2} \Rightarrow x^2 = 1 \Rightarrow \boxed{x = \pm 1}$

② f & f' DNE for $\boxed{x = 0}$ critical points are $\boxed{x = 0, 1, -1}$

5) Let $f(x) = x^3 - 6x - 4$.

a. Find all critical points of $f(x)$ (3pts)

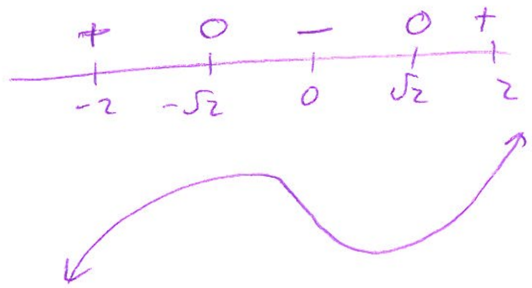
$f' = 3x^2 - 6 = 0 \Rightarrow x^2 = 2 \Rightarrow \boxed{x = \pm\sqrt{2}}$ → Critical Points

f & f' are both polynomials thus exist everywhere

b. Classify the critical points using the first derivative test (4pts) → test points around the critical points in $f'(x)$

x	f'
-2	$6 > 0$
$-\sqrt{2}$	0
0	$-6 < 0$
$\sqrt{2}$	0
2	$6 > 0$

$\Longrightarrow$



$\Rightarrow$ Max at $-\sqrt{2}$

Min at $\sqrt{2}$

Test $f''(\pm\sqrt{2})$

c. Classify the critical points again, this time using the second derivative test (3pts)

$f''(x) = 6x$

$f''(-\sqrt{2}) = -6\cdot\sqrt{2} < 0 \Rightarrow$ Concave down at $-\sqrt{2} \Rightarrow$ Max at $-\sqrt{2}$

$f''(\sqrt{2}) = 6\cdot\sqrt{2} > 0 \Rightarrow$ Concave up at $\sqrt{2} \Rightarrow$ Min at $\sqrt{2}$

More Optimization

Find two nonnegative numbers whose sum is 9 and that maximizes the product of one with the square the other.

let N_1 = a number

N_2 = another number

Objective function

Maximize $N_1 \cdot N_2^2 = Q$

Constraints

- $N_1 \geq 0$ and $N_2 \geq 0$
- $N_1 + N_2 = 9 \longrightarrow \boxed{N_1 = 9 - N_2}$

plug in

Rewrite Q in terms of 1 variable

$$Q = (9 - N_2) \cdot N_2^2 = 9N_2^2 - N_2^3$$

$$Q' = 18N_2 - 3N_2^2 = 0$$

$$N_2(18 - 3N_2) = 0$$

$\boxed{N_2 = 0}$ or $18 - 3N_2 = 0$

$\Rightarrow \boxed{N_2 = 6}$

Classify

$$Q'' = 18 - 6N_2$$

$Q''(0) = 18 > 0 \Rightarrow$ min at $N_2 = 0$

$Q''(6) = 18 - 36 = -18 < 0 \Rightarrow \boxed{\text{Max at } N_2 = 6}$

$N_1 + N_2 = 9$ and $N_2 = 6 \Rightarrow \boxed{N_1 = 3}$

Monday 3/10 Announcements & Reminders

- Test 2 will take place tomorrow during recitation
- Homework solutions are posted
- Recitation instructors have your quizzes/homework to return
- I will be in my office today from 11-2:30 and from 4:30-6
- I am holding office hours this evening from 6-7pm in Ayres 110
- Project 2 will be distributed and discussed Wednesday 3/12 in class

Test 2 3/11

- 5 mult. choice
- Derivative section
- Max/min Questions (2)
- 2 optimization
- 3 "other" short answers

See practice test on blackboard

Quotient Rule
Chain rule
given

Let $f(x) = x^6$

and $g(x) = e^{x^2} + \cos(4x)$ chain rule $g(x) = e^{x^2} \cdot \cos(4x)$

$f(g(x))$ $\boxed{\text{deriv} = f'(g(x)) \cdot g'(x)}$ $f(g(x))$

①

$F(x) = [e^{x^2} + \cos(4x)]^6$

$$\boxed{\begin{aligned} &\underline{\text{note}}: \quad \frac{d}{dx} e^{x^2} = 2xe^{x^2} \\ &\frac{d}{dx} \cos(4x) = -4\sin(4x) \end{aligned}}$$

$f'(x) = 6[e^{x^2} + \cos(4x)]^5$

$\cdot \left(\underbrace{\frac{d}{dx} e^{x^2} + \cos(4x)}_{\text{exp rule \& cos rule}}\right)$

← chain rule →

$$\boxed{= 6[e^{x^2} + \cos(4x)]^5 \cdot (2xe^{x^2} - 4\sin(4x))}$$

②

$f(x) = [e^{x^2} \cdot \cos(4x)]^6$

$f'(x) = 6[e^{x^2} \cdot \cos(4x)]^5$

$\cdot \left(\underbrace{\frac{d}{dx} e^{x^2} \cdot \cos(4x)}_{\text{product rule}}\right)$

$= 6[e^{x^2} \cdot \cos(4x)]^5$

$\cdot \left(\frac{d}{dx} e^{x^2}\right)\cos(4x) + e^{x^2}\left(\frac{d}{dx}\cos(4x)\right)$

$$= \boxed{6[e^{x^2} \cdot \cos(4x)]^5 \cdot (2xe^{x^2})\cos(4x) + e^{x^2}(-4\sin(4x))}$$

Math 152 Test 2 Topics

example test 2 on 6.6

Topics:

1) Finding derivatives of functions using derivative rules
   a. I will provide the quotient rule and chain rule ONLY. You need to know the rest
   b. Be prepared to take multiple derivatives

2) Finding the equation of a tangent line to $f(x)$ at point $x = a$
   a. Follow the 3 steps specified in the notes

3) Finding critical points ① Max/min I.P. ② Discontinuities
   a. Occur where $f'(x) = 0$ and where $f'(x)$ DNE.

4) Classifying Critical points- Two Methods

   – max: $+ \; 0 \; -$ f', a, x — min: $- \; 0 \; +$ f', a, x

   a. First Derivative Test
      i. Test points on either side of each critical point to determine where $f(x)$ is increasing/decreasing to see if we have a max or min.
   b. Second Derivative Test
      i. Test the critical point in the second derivative to determine concavity at the critical point to determine if we have a max or min.

   classify

   $f''(c.p.) > 0 \Rightarrow$ min $\quad f''(c.p.) < \Rightarrow$ max

5) Concavity
   a. Determined using second derivative
      i. $f''(a) < 0 \Rightarrow f(x)$ concave down at $f(a)$
      ii. $f''(a) > 0 \Rightarrow f(x)$ concave up at $f(a)$
      iii. $f''(a) = 0 \Rightarrow$ ~~possible~~ point of inflection at $f(a)$ or $+ \; 0 \; +$ f', a, x or $- \; 0 \; -$ f', a, x
   b. To determine all values where $f(x)$ is concave up/down set $f''(x) < 0$ and $f''(x) > 0$ and solve for resulting values
      down $\quad$ up

6) Knowing what $f'(x)$ and $f''(x)$ tell us about $f(x)$
   a. $f'(x)$
      i. Tells us the instantaneous rate of change of $f(x)$ aka the slope of the tangent line at point x. aka "steepness" of the graph at point x
      ii. Has units $\frac{units\ of\ f(x)}{units\ of\ x}$

b. $f''(x)$

i. Tells us how the first derivative is changing. That is, tells us the rate of change of the rate of change. This can be used to determine concavity

ii. Has units $\dfrac{\frac{units\ of\ f(x)}{units\ of\ x}}{units\ of\ x} = \boxed{\dfrac{units\ of\ f(x)}{(units\ of\ x)^2}}$

7) Finding the max/min of $f(x)$ on a closed interval.
   a. Find all critical points
   b. Test critical points found in the interval **&** end points of the interval in $f(x)$
   c. Choose the largest/smallest as your max/min

8) Sketching $f(x)$ given info about $f'(x)$

9) Maximization/Minimization Problems

Good Practice Problems:

- All examples from class
- All examples in each chapter (these are good because they have detailed answers)
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Ch 18:

- 18.3, 18.4, 18.5

Ch 19

- All problems from 19.1 through 19.6
- 19.12

Ch 20

- 20.1
- 20.2
- 20.3
- 20.4
- 20.5
- 20.7
- 20.14
- 20.16
- 20.18 and 20.19

20.1 (h) $f(x) = \dfrac{x}{x^2-1}$

$$f'(x) = \frac{\left(\frac{d}{dx}x\right)(x^2-1) - x\left(\frac{d}{dx}x^2-1\right)}{(x^2-1)^2} = \frac{1\cdot(x^2-1) - x(2x)}{(x^2-1)^2} = \frac{-x^2-1}{(x^2-1)^2}$$

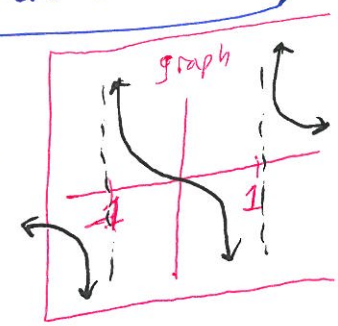
Quotient rule

$f'(x) = 0$ when $-x^2-1=0 \Rightarrow x^2=-1 \rightarrow$ $\boxed{\text{not possible}}$ (no real solutions)

$x = \pm\sqrt{-1}$

$f(x)/f'(x)$ DNE when $x^2-1=0 \Rightarrow \boxed{x=\pm 1}$

$\boxed{\text{classify :}}$ since $x=\pm 1$ came from $f(x)$ DNE, they are discontinuities



$\boxed{\text{Concavity:}}$ $$f''(x) = \frac{(-2x)(x^2-1)^2 - (-x^2-1)(2(x^2-1)^1\cdot 2x)}{((x^2-1)^2)^2}$$

Quotient rule

$$= \frac{(-2x)(x^4-2x^2+1) - (4x)(-x^4+1)}{(x^2-1)^4} = \frac{-2x^5+4x^3-2x+4x^5-4x}{(x^2-1)^4}$$

$$= \frac{2x^5+4x^3-6x}{(x^2-1)^4} = \frac{\cancel{(x^2-1)}(2x^3+6x)}{(x^2-1)^{\cancel{4}}} = \frac{2x^3+6x}{(x^2-1)^3} = \frac{2x(x^2+3)}{(x^2-1)^3}$$

This is always positive

for $f''<0$ we need $2x<0$ and $x^2-1>0$ (or) $2x>0$ and $x^2-1<0$

$\Rightarrow \boxed{x<0}$ and $\boxed{x>1 \text{ or } x<-1}$ $\Rightarrow x>0$ and $-1<x<1$

$\Rightarrow \boxed{x<-1}$ $\Longrightarrow \boxed{0<x<1}$

concave down here

$f''>0$ when $2x>0$ and $x^2-1>0$ (or) $2x<0$ and $x^2-1<0$

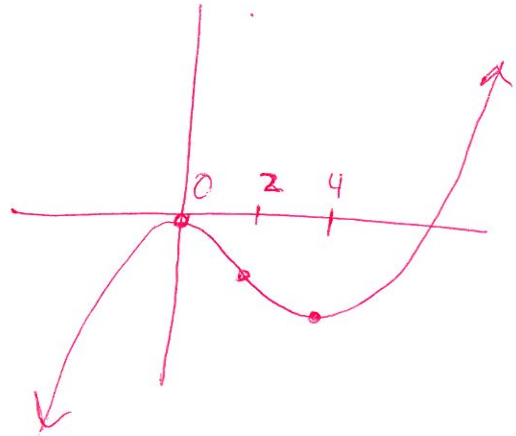
$\Rightarrow x>0$ and $x<-1$, $x>1$ $\Rightarrow x<0$ and $-1<x<1$

$\Rightarrow \boxed{x>1}$ $\Longrightarrow \boxed{-1<x<0}$

concave up here

$f''=0$ for $\boxed{x=0 \Rightarrow \text{pt of inflation}}$

Let $f(x) = x^3 - 6x^2$



$f'(x)$ tells us increasing/decreasing

$f''(x)$ tells us concave up/down

Can find/classify max at $x=0$ and min at $x=4$ by solving $f'=0$ and using 1st/2nd Deriv. test.

Where $f(x)$ increasing?

$f' = 3x^2 - 12x = 3x(x-4) \to$ need $f'>0$ for inc.
<0 for dec.

$3x(\cancel{x}-4) > 0$ if $3x>0$ and $x-4>0$ or $3x<0$, $x-4<0$

$\Rightarrow$ $x>0$ and $\boxed{x>4}$ or $\boxed{x<0}$ and $x<4$

decreasing?

$3x \cdot (x-4) < 0$ if $3x<0$ and $x-4>0$ or $3x>0$ $x-4<0$

$\Rightarrow$ $\underbrace{x<0 \text{ and } x>4}_{\text{not possible}}$ $\boxed{x>0 \text{ and } x<4}$ $\to$ decreasing

Concave up/down

$f'' = 6x-12$

$f''>0 \to 6x-12>0 \Rightarrow x>2 \to$ concave up here.

$f''<0 \to 6x-12<0 \Rightarrow x<2 \to$ concave down here

~~19.6 (b)~~ $\cancel{f(x) = (x^2+4x+2)^5}$ ~~→ use chain rule w/ "$f(x)$" $= x^5$ & "$g(x)$" $= x^2+4x+2$~~

$\cancel{f'(x) = \underbrace{5(x^2+4x+2)^4}_{\text{"}f'(g(x))\text{"}}\underbrace{(2x+4)}_{g'(x)}}$ ~~⟹ "$f'(x)$" $= 5x^4$~~

~~⟹ "$f'(g(x))$" $= 5(x^2+4x+2)^4$~~

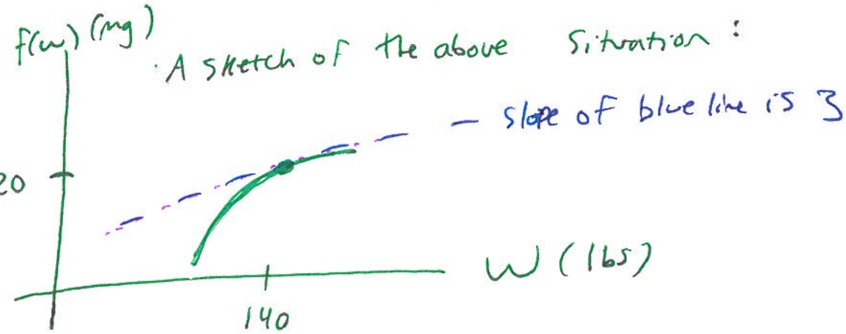
~~and "$g'(x)$" $= 2x+4$~~

19.12 (a) $f(140) = 120$ means a 140 lbs person needs 120 mg of painkiller

$\boxed{f'(140) = 3}$ means at $f(140)$ the slope of the graph is $3 \frac{mg}{lbs}$

ie, every lb increase requires 3 mg extra of painkiller

A sketch of the above situation:



(b) Estimate $f(145)$ → amount of painkillers in mg needed for a 145 lb patient

145 is 5 more than 140

$5 \text{ lb increase} \cdot 3 \frac{mg}{lb} =$ 15 milligrams more than needed at 140 lbs.

$f(140) = 120$

So $f(145) \approx 120 + 15 = 135$ mg

just an estimate

6) Optimization: Show as much work as you can to receive partial credit. (10 points each)

a) Find two numbers whose difference is 9 that ~~maximize~~ Minimize the sum of their squares

Variables

$N_1 =$ a number

$N_2 =$ another number

Constraints

$N_1 - N_2 = 9 \Rightarrow \boxed{N_1 = 9 + N_2}$ or $-N_2 = 9 - N_1$

Objective Function

~~Max~~ min. $N_1^2 + N_2^2$ → rewrite w/ only one variable using

~~Max~~ min $(9+N_2)^2 + N_2^2 = N_2^2 + 18N_2 + 81 + N_2^2$

"Q" $= 2N_2^2 + 18N_2 + 81$ → find the ~~Max~~ min

$Q' = 4N_2 + 18 = 0$

$\boxed{N_2 = -4.5}$ → verify its a min using 2nd deriv test.

$N_1 - N_2 = 9 \Rightarrow N_1 - (-4.5) = 9 \Rightarrow \boxed{N_1 = 4.5}$