1.1- Real Numbers

The following material found on this page is assumed knowledge that can be used as a reference and reminder. For more help reviewing fractions or any of the following properties see the information on pages 3-9 of the textbook and/or come see me.

PROPERTIES OF REAL NUMBERS

Property	Example	Description
Commutative Properties		
$a + b = b + a$	$7 + 3 = 3 + 7$	When we add two numbers, order doesn't matter.
$ab = ba$	$3 \cdot 5 = 5 \cdot 3$	When we multiply two numbers, order doesn't matter.
Associative Properties		
$(a + b) + c = a + (b + c)$	$(2 + 4) + 7 = 2 + (4 + 7)$	When we add three numbers, it doesn't matter which two we add first.
$(ab)c = a(bc)$	$(3 \cdot 7) \cdot 5 = 3 \cdot (7 \cdot 5)$	When we multiply three numbers, it doesn't matter which two we multiply first.
Distributive Property		
$a(b + c) = ab + ac$ $(b + c)a = ab + ac$	$2 \cdot (3 + 5) = 2 \cdot 3 + 2 \cdot 5$ $(3 + 5) \cdot 2 = 2 \cdot 3 + 2 \cdot 5$	When we multiply a number by a sum of two numbers, we get the same result as we would get if we multiply the number by each of the terms and then add the results.

PROPERTIES OF FRACTIONS

Property	Example	Description
1. $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$	$\frac{2}{3} \cdot \frac{5}{7} = \frac{2 \cdot 5}{3 \cdot 7} = \frac{10}{21}$	When **multiplying fractions**, multiply numerators and denominators.
2. $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$	$\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \cdot \frac{7}{5} = \frac{14}{15}$	When **dividing fractions**, invert the divisor and multiply.
3. $\frac{a}{c} + \frac{b}{c} = \frac{a + b}{c}$	$\frac{2}{5} + \frac{7}{5} = \frac{2 + 7}{5} = \frac{9}{5}$	When **adding fractions** with the **same denominator**, add the numerators.
4. $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$	$\frac{2}{5} + \frac{3}{7} = \frac{2 \cdot 7 + 3 \cdot 5}{35} = \frac{29}{35}$	When **adding fractions** with **different denominators**, find a common denominator. Then add the numerators.
5. $\frac{ac}{bc} = \frac{a}{b}$	$\frac{2 \cdot 5}{3 \cdot 5} = \frac{2}{3}$	**Cancel** numbers that are **common factors** in numerator and denominator.
6. If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$	$\frac{2}{3} = \frac{6}{9}$, so $2 \cdot 9 = 3 \cdot 6$	**Cross-multiply**.

PROPERTIES OF NEGATIVES

Property	Example
1. $(-1)a = -a$	$(-1)5 = -5$
2. $-(-a) = a$	$-(-5) = 5$
3. $(-a)b = a(-b) = -(ab)$	$(-5)7 = 5(-7) = -(5 \cdot 7)$
4. $(-a)(-b) = ab$	$(-4)(-3) = 4 \cdot 3$
5. $-(a + b) = -a - b$	$-(3 + 5) = -3 - 5$
6. $-(a - b) = b - a$	$-(5 - 8) = 8 - 5$

DEFINITION OF ABSOLUTE VALUE

If a is a real number, then the **absolute value** of a is

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

PROPERTIES OF ABSOLUTE VALUE

Property	Example	Description
1. $\lvert a \rvert \geq 0$	$\lvert -3 \rvert = 3 \geq 0$	The absolute value of a number is always positive or zero.
2. $\lvert a \rvert = \lvert -a \rvert$	$\lvert 5 \rvert = \lvert -5 \rvert$	A number and its negative have the same absolute value.
3. $\lvert ab \rvert = \lvert a \rvert \lvert b \rvert$	$\lvert -2 \cdot 5 \rvert = \lvert -2 \rvert \lvert 5 \rvert$	The absolute value of a product is the product of the absolute values.
4. $\left\lvert \frac{a}{b} \right\rvert = \frac{\lvert a \rvert}{\lvert b \rvert}$	$\left\lvert \frac{12}{-3} \right\rvert = \frac{\lvert 12 \rvert}{\lvert -3 \rvert}$	The absolute value of a quotient is the quotient of the absolute values.

Types of Numbers

- Natural Numbers
- Integers
- Rational Numbers

- Irrational Numbers
- Real Numbers

Sets and Intervals

Set-

Two ways to represent a set:

1)

2)

Notation:

Intervals

1.2- Exponents & Radicals

EXPONENTIAL NOTATION

If a is any real number and n is a positive integer, then the **nth power** of a is

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$$

The number a is called the **base**, and n is called the **exponent**.

Properties of Exponents

ZERO AND NEGATIVE EXPONENTS

If $a \neq 0$ is any real number and n is a positive integer, then

$$a^0 = 1 \qquad \text{and} \qquad a^{-n} = \frac{1}{a^n}$$

EXs:

LAWS OF EXPONENTS

Law	Example	Description
1. $a^m a^n = a^{m+n}$	$3^2 \cdot 3^5 = 3^{2+5} = 3^7$	To multiply two powers of the same number, add the exponents.
2. $\frac{a^m}{a^n} = a^{m-n}$	$\frac{3^5}{3^2} = 3^{5-2} = 3^3$	To divide two powers of the same number, subtract the exponents.
3. $(a^m)^n = a^{mn}$	$(3^2)^5 = 3^{2 \cdot 5} = 3^{10}$	To raise a power to a new power, multiply the exponents.
4. $(ab)^n = a^n b^n$	$(3 \cdot 4)^2 = 3^2 \cdot 4^2$	To raise a product to a power, raise each factor to the power.
5. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	$\left(\frac{3}{4}\right)^2 = \frac{3^2}{4^2}$	To raise a quotient to a power, raise both numerator and denominator to the power.

$$(8a^2z)\left(\frac{1}{2}a^3z^4\right) =$$

$$\frac{\left(u^{-1}v^2\right)^2}{(u^3v^{-2})^3} =$$

$$\left(\frac{xy^{-2}z^{-3}}{x^2y^3z^{-4}}\right)^{-3}$$

$$(2u^2v^3)^3(3u^{-3}v)^2$$

Scientific Notation

Some numbers that humans encounter are too large or too small to write out in decimal form, which is why scientific notation was developed.

SCIENTIFIC NOTATION

A positive number x is said to be written in **scientific notation** if it is expressed as follows:

$$x = a \times 10^n \qquad \text{where } 1 \le a < 10 \text{ and } n \text{ is an integer}$$

Examples:

Radicals

Definition of square root:

$$\sqrt{a} = b \qquad \text{means} \qquad b^2 = a \qquad \text{and} \qquad b \ge 0$$

DEFINITION OF nth ROOT

If n is any positive integer, then the **principal nth root** of a is defined as follows:

$$\sqrt[n]{a} = b \qquad \text{means} \qquad b^n = a$$

If n is even, we must have $a \ge 0$ and $b \ge 0$.

PROPERTIES OF nth ROOTS

Property

1. $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$

2. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

3. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

4. $\sqrt[n]{a^n} = a \quad$ if n is odd

5. $\sqrt[n]{a^n} = |a| \quad$ if n is even

Examples

Simplifying Radical Expressions- Method #1

One way to simplify a radical expression is to factor out the largest nth root and combine with like radicals

Exs:

DEFINITION OF RATIONAL EXPONENTS

For any rational exponent m/n in lowest terms, where m and n are integers and $n > 0$, we define

$$a^{m/n} = (\sqrt[n]{a})^m \qquad \text{or equivalently} \qquad a^{m/n} = \sqrt[n]{a^m}$$

If n is even, then we require that $a \geq 0$.

$$a^{1/n} = \sqrt[n]{a}$$

EXs:

Simplifying Radical Expressions- Method #2

A second method to simplify a radical expression is to change the radical to an exponent and use the properties from page 1.

Exs:

Rationalizing the Denominator

If a fraction has a radical in the denominator of the form $\sqrt[n]{a^m}$ we can rationalize it by multiplying both the numerator and denominator by $\sqrt[n]{a^{n-m}}$

Exs:

1.3- Algebraic Expressions

POLYNOMIALS

A **polynomial** in the variable x is an expression of the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n$ are real numbers, and n is a nonnegative integer. If $a_n \neq 0$, then the polynomial has **degree** $\boldsymbol{n}$. The monomials $a_k x^k$ that make up the polynomial are called the **terms** of the polynomial.

Ex: A degree 4 polynomial-

NonEx:

Adding and Subtracting Polynomials

Like terms: Terms with the same variable raised to the same power

When adding/subtracting two polynomials, you can only combine like terms

Ex: $(x^3 + 2x - 5) - (4x^3 - x^2 - 4x + 10)$

Multiplying Algebraic Expressions

To find the product of two algebraic expressions you must use the **distributive property** to multiply each term of the first expression by each term of the second expression.

Ex: $(4y - 10)(6y^2 + 8y + 2)$

Ex:

Special Product Formulas

SPECIAL PRODUCT FORMULAS

If A and B are any real numbers or algebraic expressions, then

1. $(A + B)(A - B) = A^2 - B^2$ — Sum and product of same terms
2. $(A + B)^2 = A^2 + 2AB + B^2$ — Square of a sum
3. $(A - B)^2 = A^2 - 2AB + B^2$ — Square of a difference
4. $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$ — Cube of a sum
5. $(A - B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$ — Cube of a difference

Ex: $(4 - 5x)^2$

Ex:

Ex: $(x^2 + y)^3$

Ex:

Factoring

The rest of the section is devoted to factoring polynomials. Factoring is the opposite of multiplying algebraic expressions. That is, **factoring** an expression is re-writing the expression as a product of simpler ones.

<u>Approaches to Factoring:</u>

1) Look for a common factor
2) "Trial and Error" (with 3 terms)
3) Use a Special Factoring Formula
4) Grouping Terms (with 4 or more terms)

1) Looking for a common factor

If each term of a polynomial share a common factor you can divide each term by the common factor and put the common factor out in front of the resulting expression

Ex: $4x^2 + 2x - 8$

Common factor:

Ex: $8x^4y^2 + 6x^3y^3 - 2xy^4$

Common factor:

2) Trial and Error

Trinomial- a polynomial of the form $ax^2 + bx + c$ where *a, b and c* are any real number

Trial and error is a method used to factor a trinomial

To factor such a polynomial you have to think of two numbers (call them *r and s)* such that *r+s=b and rs=c.* Then, factor the expression using the form $(x + r)(x + s)$

Ex: $x^2 + x - 6$

Ex: $x^2 - 2x - 35$

If $a \neq 1$ then we need to find numbers *p, q, r and s* such that *pq=a, rs=c* and *ps + qr = b* . Then, factor the expression using the form $(px + r)(qx + s)$

Ex: $6x^2 + 7x - 5$

3) Special Factoring Formulas

SPECIAL FACTORING FORMULAS

Formula	Name
1. $A^2 - B^2 = (A - B)(A + B)$	Difference of squares
2. $A^2 + 2AB + B^2 = (A + B)^2$	Perfect square
3. $A^2 - 2AB + B^2 = (A - B)^2$	Perfect square
4. $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$	Difference of cubes
5. $A^3 + B^3 = (A + B)(A^2 - AB + B^2)$	Sum of cubes

Ex: $(4 + y)^3 + 8$

Ex: $27 - x^3$

Ex

4) Grouping Terms

If a polynomial has at least 4 terms, first grouping terms with a common factor can be an effective approach. Then factor each grouping and look for a final common factor.

Ex: $x^3 + x^2 + 4x + 4$

Ex: $x^3 + 2x^2 - 6x - 12$

1.4- Rational Expressions

Domain of an Algebraic Expression

Rational Expression- a fractional expression where both the numerator as well as the denominator are polynomials

Exs: $\frac{2x}{x-1}$ $\frac{x}{x^2+1}$ $\frac{x^3-x}{x^2-5x+6}$

Domain of an Algebraic Expression- All real numbers that can be plugged into the given variable that will result in a sound expression.

To find out what the domain of an expression is consider the real numbers that, when plugged into the expression, will result in an undefined value.

We write the domain in set-builder notation

Possible problems: **a)** Dividing by 0 **b)** Taking the root of a negative number **c)** Both **a** and **b**

Ex: $\frac{x}{(x^2-4)}$ Ex: $\frac{\sqrt{r}}{(r+9)(r-12)}$ Ex: $y^2+4y-10$

Simplifying Rational Expressions

In order to simplify rational expressions you can factor the numerator and denominator and cancel using the following property:

$$\boxed{\frac{AC}{BC}=\frac{A}{B}}$$

Ex: $\frac{x+2}{(x^2-6x-16)}$ Ex: $\frac{x^2-2x-3}{(x^2+6x+5)}$

Multiplying Rational Expressions

When two rational expressions are being multiplied use the following property:

$$\boxed{\frac{A}{B}\cdot\frac{C}{D}=\frac{AC}{BD}}$$

Ex: $\dfrac{P-6}{(P^2+2P-1)}\cdot\dfrac{4+P^2}{P}$

Dividing Rational Expressions

When dividing one rational expression by another use the following property combined with the above property of multiplication

That is, use $\boxed{\frac{A}{B}\div\frac{C}{D}=\frac{A}{B}\cdot\frac{D}{C}}$ followed by $\boxed{\frac{A}{B}\cdot\frac{C}{D}=\frac{AC}{BD}}$

Ex: $\dfrac{x-2}{x^2+4}\div\dfrac{x^3+1}{x+6}$

Ex: $\dfrac{z+10}{1-z^2}\div\dfrac{10+z}{z}$

Another?

Adding and Subtracting Rational Expression

- In order to add or subtract two rational expressions, like any fraction, they must have the same denominator.
- To write each term with the same denominator you must determine a common multiple of both denomiators.
- The easiest way to find a common multiple is to factor each denominator and take the product of the **distinct** factors. That is, multiply together each non-repeated factor of each denominator.
- Once you have determined the common multiple, multiply each rational expression appropriately by a form of 1 to reach a common denominator.
- Once both expressions have a common denominator, add/subtract the numerators while keeping the denominator the same

Ex: $\frac{4}{x-3} - \frac{x}{x+7}$

Ex: $\frac{1}{t^2-4} + \frac{t}{(t-2)^2}$

Another?

Compound Fractions- A fraction where the numerator, the denominator or both are themselves fractional expressions

2 methods: **a)** Combine the numerator and/or denominator and then invert the fraction

b) Multiply the numerator and denominator by a common factor

$$\frac{x+\frac{1}{x+2}}{x-\frac{1}{x+2}}$$

Method a:

Method b:

Ex: $\dfrac{\frac{t}{p}-\frac{p}{t}}{\frac{1}{t^2}+\frac{1}{p^2}}$

Rationalizing the Denominator or the Numerator

If the numerator or denominator of a fraction has the form $A + B\sqrt{C}$ we can rationalize the numerator or denominator by multiplying the fraction by $\frac{A-B\sqrt{C}}{A-B\sqrt{C}}$ (called the <u>conjugate radical)</u>

If the numerator or denominator of a fraction has the form $A - B\sqrt{C}$ we can rationalize the numerator or denominator by multiplying the fraction by $\frac{A+B\sqrt{C}}{A+B\sqrt{C}}$

We are able to do this because of the <u>special product formulas</u> from section 1.3. Using these formulas we do not have to do any of the algebra

Ex: $\frac{5}{2+\sqrt{5}}$

Ex: $\frac{2\sqrt{3}-4}{x}$

<u>Don't make these common mistakes:</u>

Correct multiplication property	Common error with addition
$(a \cdot b)^2 = a^2 \cdot b^2$	$(a+b)^2 \neq a^2 + b^2$
$\sqrt{a \cdot b} = \sqrt{a}\sqrt{b} \quad (a, b \geq 0)$	$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$
$\sqrt{a^2 \cdot b^2} = a \cdot b \quad (a, b \geq 0)$	$\sqrt{a^2 + b^2} \neq a + b$
$\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{a \cdot b}$	$\frac{1}{a} + \frac{1}{b} \neq \frac{1}{a+b}$
$\frac{ab}{a} = b$	$\frac{a+b}{a} \neq b$
$a^{-1} \cdot b^{-1} = (a \cdot b)^{-1}$	$a^{-1} + b^{-1} \neq (a+b)^{-1}$

1.5- Equations

Linear Equation- an equation of the form $ax + b = 0$ where $a, b \epsilon \mathbb{R}$

Quadratic Equation- an equation of the form $ax^2 + bx + c = 0$ where $a, b, c \epsilon \mathbb{R}$

In order to solve an equation you must isolate the variable on one side of the equal sign. That is, you have to "get the variable by itself."

The general method for solving equations is to reverse the order of operations. When doing so, as long as you carry out the same operation on both sides of the equal sign you will preserve the equality and get the correct answer.

Key Facts: **1)** If $\sqrt{x} = y \; then \; x = y^2$ **2)** If $x = y^2 then \; y = \pm\sqrt{x}$

Solving Linear Equations

To solve a linear equation you must get all variables on one side of the equal sign and all numbers on the other side and then multiply/divide as needed to isolate the variable.

Try this one yourself: $2x + 8 = 6 - 4x$

Solving Quadratic Equations

Methods: 1) Take the square root

2) Complete the square

3) Quadratic Formula

1) If the quadratic equation is simple enough you may just need to take the square root of both sides

Ex: $y^2 = 81$

Ex: $(y + 4)^2 = 10$

Ex:

2)

COMPLETING THE SQUARE

To make $x^2 + bx$ a perfect square, add $\left(\frac{b}{2}\right)^2$, the square of half the coefficient of x. This gives the perfect square

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

Ex: $2v^2 - 6v + 4 = 0$

Ex: $y^2 + 3y - 12 = 0$

One more?

3) Quadratic Formula- Can be used to solve <u>any</u> quadratic equation

THE QUADRATIC FORMULA

The roots of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Ex: $2v^2 - 6v + 4 = 0$

Ex: $y^2 + 3y - 12 = 0$

Ex:

Discriminant

THE DISCRIMINANT

The **discriminant** of the general quadratic $ax^2 + bx + c = 0$ $(a \neq 0)$ is $D = b^2 - 4ac$.

1. If $D > 0$, then the equation has two distinct real solutions.
2. If $D = 0$, then the equation has exactly one real solution.
3. If $D < 0$, then the equation has no real solution.

Ex: $2v^2 - 6v + 4 = 0$

Ex: $y^2 + 3y - 12 = 0$

Ex:

Solving for a variable in terms of other variables

You can use the same steps as above to solve an equation for a given variable

Ex: solve the following equation for the variable m : $G\frac{mV}{r^2} = F$

Solving an equation with a fractional expression

First multiply both equations by the least common denominator. Doing so will eliminate all fractional expression(s).

Once you have eliminated the fractional expression(s) you can solve the equation using one of the methods above

Ex: $\frac{1}{x-1} - \frac{2}{x} = 1$

Solving an equation with a radical expression

First isolate the radical.

Second eliminate the radical by raising both sides of the equation to the same power.

Finally, solve the equation as you would any other.

Ex: $2x + \sqrt{x+1} = 8$

Solving an equation with an absolute value

Property: If $|x| = a$ then $x = a$ or $x = -a$

$|14 - x| = 28$

$1 = |x - 6|$

1.7- Inequalities

RULES FOR INEQUALITIES

Rule	Description
1. $A \le B \quad \Leftrightarrow \quad A + C \le B + C$	**Adding** the same quantity to each side of an inequality gives an equivalent inequality.
2. $A \le B \quad \Leftrightarrow \quad A - C \le B - C$	**Subtracting** the same quantity from each side of an inequality gives an equivalent inequality.
3. If $C > 0$, then $A \le B \quad \Leftrightarrow \quad CA \le CB$	**Multiplying** each side of an inequality by the same *positive* quantity gives an equivalent inequality.
4. If $C < 0$, then $A \le B \quad \Leftrightarrow \quad CA \ge CB$	**Multiplying** each side of an inequality by the same *negative* quantity *reverses the direction* of the inequality.
5. If $A > 0$ and $B > 0$, then $A \le B \quad \Leftrightarrow \quad \frac{1}{A} \ge \frac{1}{B}$	**Taking reciprocals** of each side of an inequality involving *positive* quantities *reverses the direction* of the inequality.
6. If $A \le B$ and $C \le D$, then $A + C \le B + D$	Inequalities can be added.

Make sure you know and understand #3, #4 and #5

Linear Inequalities

If an inequality is linear (the highest power of any variable is 1), you can use algebra and the above rules to isolate variable. Just make sure every operation you do you carry it out on all parts of the inequality.

$8 < -5x + 3 \le 18$

$2x \ge 8x - 6 > 2x + 12$

Nonlinear Inequalities

GUIDELINES FOR SOLVING NONLINEAR INEQUALITIES

1. **Move All Terms to One Side.** If necessary, rewrite the inequality so that all nonzero terms appear on one side of the inequality sign. If the nonzero side of the inequality involves quotients, bring them to a common denominator.
2. **Factor.** Factor the nonzero side of the inequality.
3. **Find the Intervals.** Determine the values for which each factor is zero. These numbers will divide the real line into intervals. List the intervals that are determined by these numbers.
4. **Make a Table or Diagram.** Use test values to make a table or diagram of the signs of each factor on each interval. In the last row of the table determine the sign of the product (or quotient) of these factors.
5. **Solve.** Determine the solution of the inequality from the last row of the sign table. Be sure to check whether the inequality is satisfied by some or all of the endpoints of the intervals. (This may happen if the inequality involves $\leq$ or $\geq$.)

$T^2 > 3(T+6)$

$(y-3)(y+6)^2 < 0$

Another?

Solving an Inequality Involving a Quotient

Follow the exact same steps as above

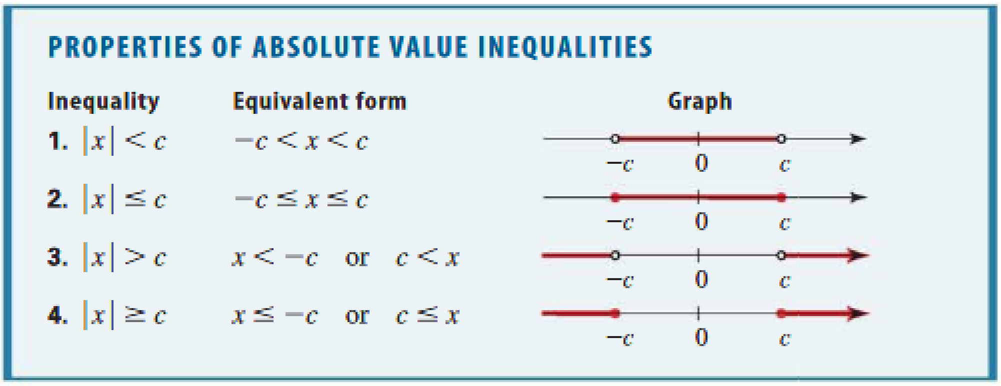
The main difference is that you need to factor both the numerator as well as the denominator and consider all factors of both in order to create your intervals.

Note: Any interval end-point found in the denominator will **NOT** satisfy the inequality

$\frac{2x+1}{x-5} \leq 3$

$\frac{s+1}{s-3} > -2$

Absolute Value Inequalities



PROPERTIES OF ABSOLUTE VALUE INEQUALITIES

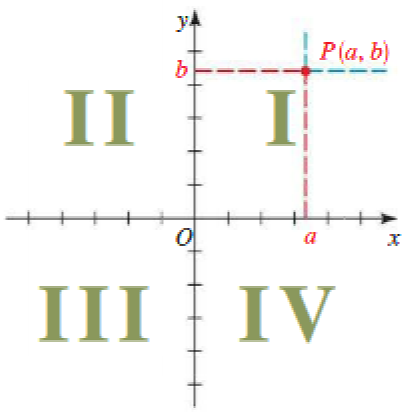
Inequality	Equivalent form	Graph
1. $\lvert x \rvert < c$	$-c < x < c$	$-c$ 0 c
2. $\lvert x \rvert \le c$	$-c \le x \le c$	$-c$ 0 c
3. $\lvert x \rvert > c$	$x < -c$ or $c < x$	$-c$ 0 c
4. $\lvert x \rvert \ge c$	$x \le -c$ or $c \le x$	$-c$ 0 c

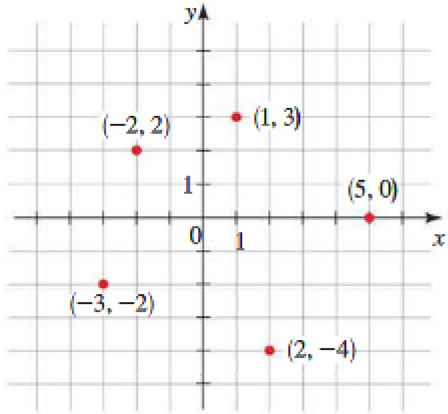
$|4x + 6| \le 18$

$18 < |4x + 6|$

1.8- Coordinate Geometry

The Coordinate Plane / Cartesian Plane / X-Y Plane





Graphing Regions in the Coordinate Plane

Describe the following regions shown in set-builder notation in words, think about what it means and then graph them in the coordinate plane.

$\{\ (x, y)\ \mid\ y \geq 0\ \}$

$\{\ (x, y)\ \mid\ \ |x| \geq 0\ \}$

Ex:

Distance Formula

DISTANCE FORMULA

The distance between the points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the plane is

$$d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint Formula

MIDPOINT FORMULA

The midpoint of the line segment from $A(x_1, y_1)$ to $B(x_2, y_2)$ is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Find the distance between (-2 , 5) and (10 , 0)

Find the Midpoint of the line segment joining (-2 , 5) and (10 , 0)

Graphs of Equations with Two Variables

One of the easiest ways to graph any equation with two variables is to create a table.

The columns of the table will be x values, y values and the resulting point (x,y)

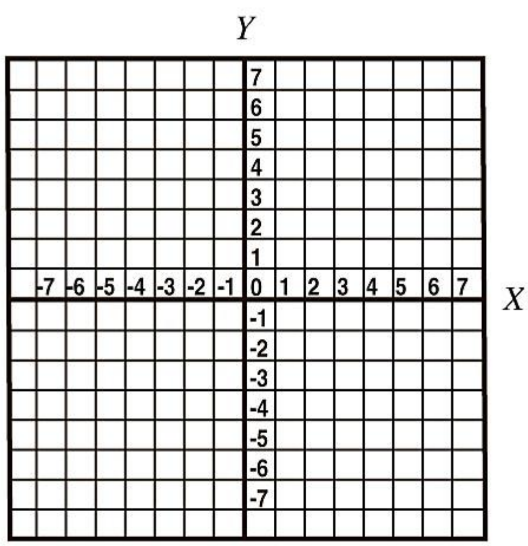
You can do this with any two variable equation

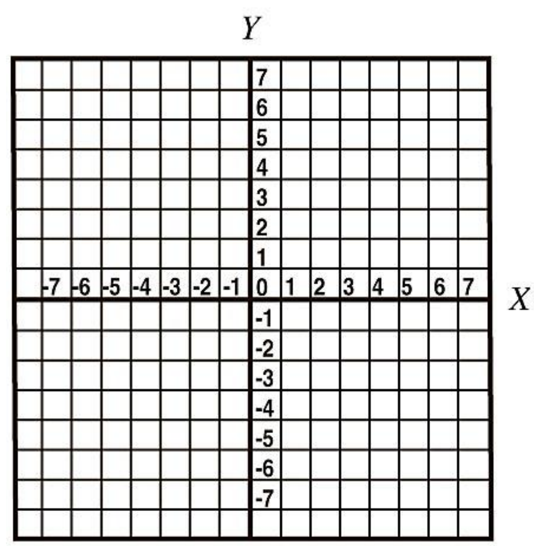
Use a table to Graph $y = |x|$

x	y = |x|	(x,y)

Use a table to Graph $y = 4x^2 - 5$

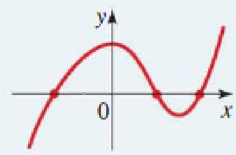
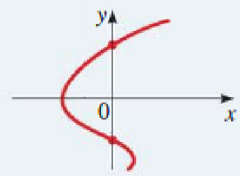
x	$y = 4x^2 - 5$	(x,y)



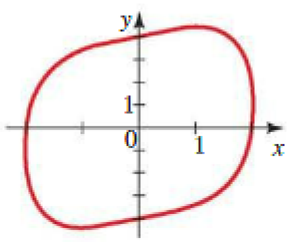


Intercepts

DEFINITION OF INTERCEPTS

Intercepts	How to find them	Where they are on the graph
***x*-intercepts:** The x-coordinates of points where the graph of an equation intersects the x-axis	Set $y = 0$ and solve for x	
***y*-intercepts:** The y-coordinates of points where the graph of an equation intersects the y-axis	Set $x = 0$ and solve for y	

Find the x and y intercept of the following graph



Find the x and y intercept of the following equation

$x^4 + y^2 - xy = 16$

Ex:

Circles

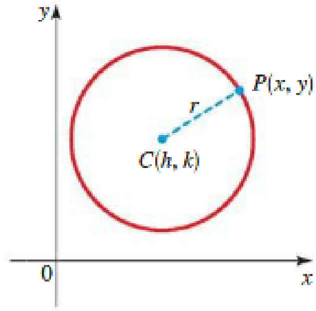
EQUATION OF A CIRCLE

An equation of the circle with center (h, k) and radius r is

$$(x - h)^2 + (y - k)^2 = r^2$$

This is called the **standard form** for the equation of the circle. If the center of the circle is the origin $(0, 0)$, then the equation is

$$x^2 + y^2 = r^2$$

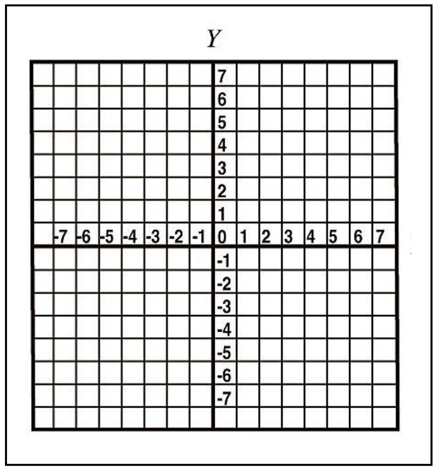


Find the center and radius of the following circle and sketch the graph

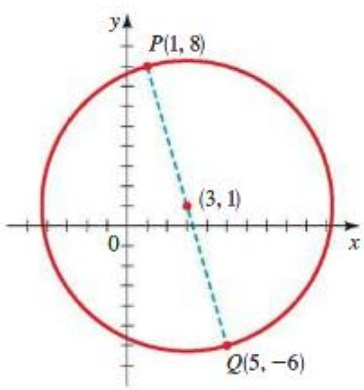
$(x + 1)^2 + (y + 2)^2 = 36$

Center:

Radius:

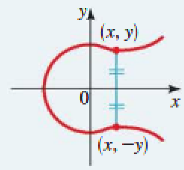
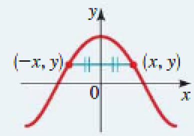
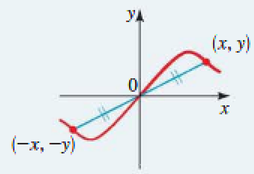


Find the equation of a circle whose diameter has endpoints (1,8) and (5,-6)



Symmetry

DEFINITION OF SYMMETRY

Type of symmetry	How to test for symmetry	What the graph looks like (figures in this section)	Geometric meaning
Symmetry with respect to the x-axis	The equation is unchanged when y is replaced by $-y$	 (Figures 13, 18)	Graph is unchanged when reflected in the x-axis
Symmetry with respect to the y-axis	The equation is unchanged when x is replaced by $-x$	 (Figures 9, 10, 11, 13, 17)	Graph is unchanged when reflected in the y-axis
Symmetry with respect to the origin	The equation is unchanged when x is replaced by $-x$ and y by $-y$	 (Figures 13, 19)	Graph is unchanged when rotated 180° about the origin

If an equation has a type of symmetry it can be graphed easily using a table. You can do this by using a table to plot only positive values, which will create half of the graph. Then simply reflect the graph around the appropriate axis to create the second half of the graph.

Determine if the following equation has any type of symmetry and use this information to graph the equation

$2y - x^2 = 1$

1.10- Lines

SLOPE OF A LINE

The **slope** m of a nonvertical line that passes through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope of a vertical line is not defined.

3 Types of Equations of a Line

1)

POINT-SLOPE FORM OF THE EQUATION OF A LINE

An equation of the line that passes through the point (x_1, y_1) and has slope m is

$$y - y_1 = m(x - x_1)$$

2)

SLOPE-INTERCEPT FORM OF THE EQUATION OF A LINE

An equation of the line that has slope m and y-intercept b is

$$y = mx + b$$

3)

GENERAL EQUATION OF A LINE

The graph of every **linear equation**

$$Ax + By + C = 0 \qquad (A, B \text{ not both zero})$$

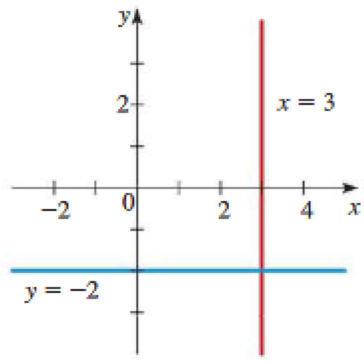
is a line. Conversely, every line is the graph of a linear equation.

Every non-vertical and non-horizontal line can be written in each of the previous forms

Vertical and Horizontal Lines

VERTICAL AND HORIZONTAL LINES

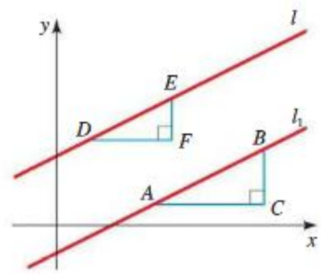
An equation of the vertical line through (a, b) is $x = a$.
An equation of the horizontal line through (a, b) is $y = b$.



Parallel Lines

PARALLEL LINES

Two nonvertical lines are parallel if and only if they have the same slope.



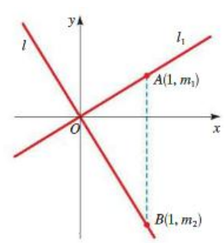
Perpendicular Lines

PERPENDICULAR LINES

Two lines with slopes m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$, that is, their slopes are negative reciprocals:

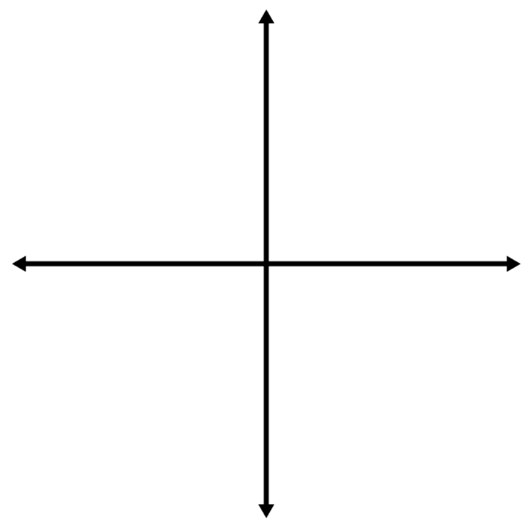
$$m_2 = -\frac{1}{m_1}$$

Also, a horizontal line (slope 0) is perpendicular to a vertical line (no slope).



1) Determine the equation of the line that has a slope of 5 and that passes through (2 , 3). Use any type of equation you would like.

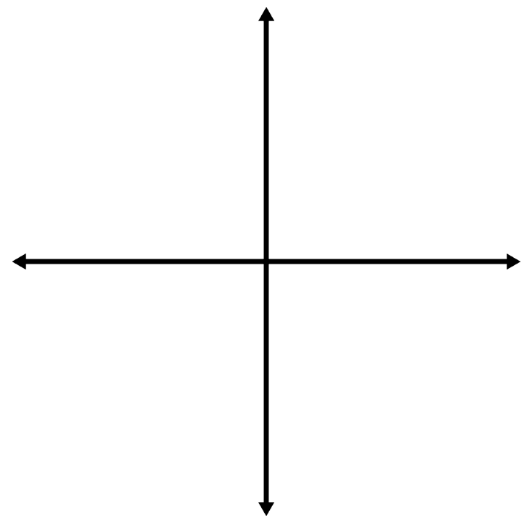
2) Determine the equation of the line passing through (2 , -5) and (-4 , 3) in <u>all 3 forms</u> and then graph the line in the coordinate plane



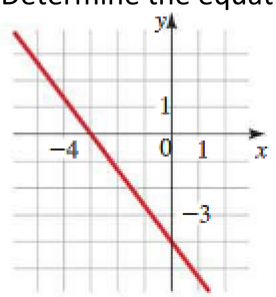
3) Determine the equation of the line in <u>point-slope</u> form that passes through (4 , 5) and has a slope of $-\frac{1}{2}$

4) Find the x-intercept and y-intercept of $-3x - 5y + 30 = 0$

5) Find the slope and y-intercept of $3x - 2y = 12$ and use them to graph the equation



6) Determine the equation of the following line and write it in <u>all 3 forms</u>



7) Determine the equation of the line that passes through $(\frac{1}{2}, -\frac{2}{3})$ and is perpendicular to $4x - 8y = 1$ in <u>slope intercept form</u>

2.1- Functions

Our world is full of objects, concepts and forces. It is obvious that the state of many of these things have an effect on one another. That is, when one changes it affects the other.

Ex: Age and height- As a person's age changes so does their height. That is, your height depends on your age

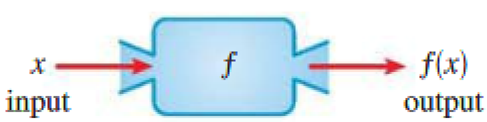
Ex: Your grade in this class and the effort you put into the course- The more effort you put in the higher your average will be

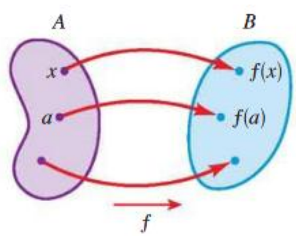
Ex:

However, simply saying one thing depends on something else does not give any specific information about how the two are related. The best way to describe a relationship between two things is to use a <u>function</u>, which is something that associates one quantity (called the <u>input</u>) with another quantity (called the <u>output</u>).

DEFINITION OF A FUNCTION

A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B.





Usually sets A and B are sets of real numbers

We call set A (seen above) the <u>domain</u>- the set of all possible values of x that can be plugged into the function. Also known as the <u>independent variable</u>

We call set B the <u>range</u>- the set of all possible values of f(x) that result from plugging in values from the domain. Also known as the <u>dependent variable.</u> In set-builder notation we can write the range as $\{f(x)|x\epsilon A\}$

Basic Example: determine the domain and range of $f(x) = x^2 + 4$ and express them in interval notation

Evaluating Functions

To evaluate a function at value simply substitute the value into the variable

Ex: If $f(x) = 3x^2 - 4x + 1$, evaluate the following

a) ***f*(3)**

b) ***f(-a)***

$$c)\ \frac{f(a+h) - f(a)}{h}$$

Piecewise Functions- a function whose output values are broken into different pieces, each of which depends on the input value

Ex: $g(x) = \begin{cases} 12 & \text{If} -\infty \leq x \leq 100 \\ x^2 & \text{If x} > 100 \end{cases}$

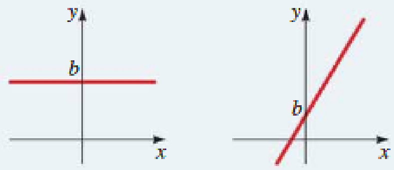
g(27.9)=

g(100,000)=

2.2- Graphs of Functions

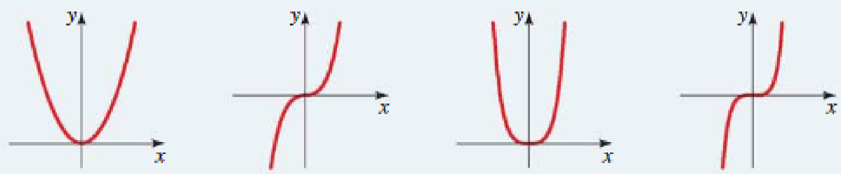
SOME FUNCTIONS AND THEIR GRAPHS

Linear functions
$f(x) = mx + b$



$f(x) = b$ $\qquad$ $f(x) = mx + b$

Power functions
$f(x) = x^n$

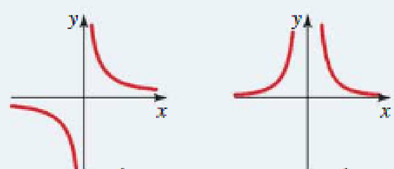


$f(x) = x^2$ $\qquad$ $f(x) = x^3$ $\qquad$ $f(x) = x^4$ $\qquad$ $f(x) = x^5$

Root functions
$f(x) = \sqrt[n]{x}$

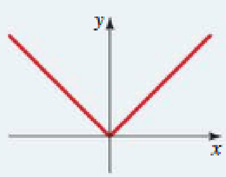
$f(x) = \sqrt{x}$ $\qquad$ $f(x) = \sqrt[3]{x}$ $\qquad$ $f(x) = \sqrt[4]{x}$ $\qquad$ $f(x) = \sqrt[5]{x}$

Reciprocal functions
$f(x) = \dfrac{1}{x^n}$



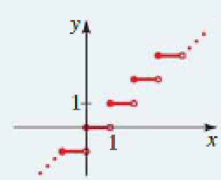
$f(x) = \dfrac{1}{x}$ $\qquad$ $f(x) = \dfrac{1}{x^2}$

Absolute value function
$f(x) = |x|$



$f(x) = |x|$

Greatest integer function
$f(x) = [\![x]\!]$



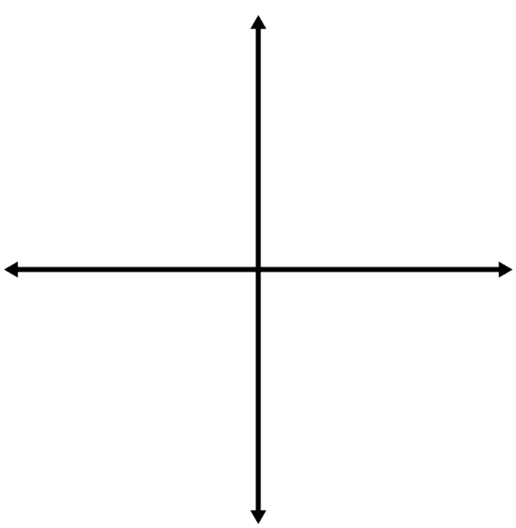
$f(x) = [\![x]\!]$

Graphing Functions by Plotting Points

Any function, especially those that are more complicated, can be ***sketched*** by making a table

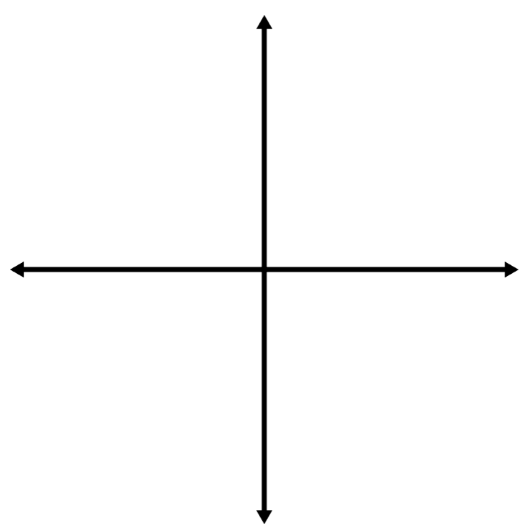
Use a table to Graph $f(x) = \sqrt{x+4}$

x	F(x)	(x,y)



Use a table to Graph $f(x) = \frac{x}{|x|}$

x	F(x)	(x,y)



Graphing Piecewise Functions

To graph a piecewise function you must consider what the domain of each piece is.

To draw the graph you can either use known information about the type of function or create a table for each piece of the function.

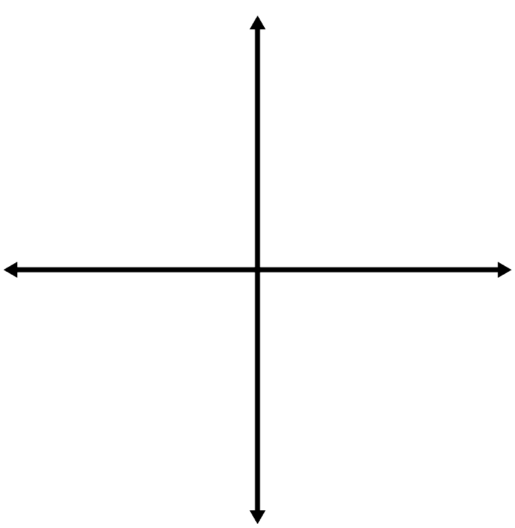
At the intersection points of the pieces, use an open circle or closed circle similar to drawing the graph of an interval.

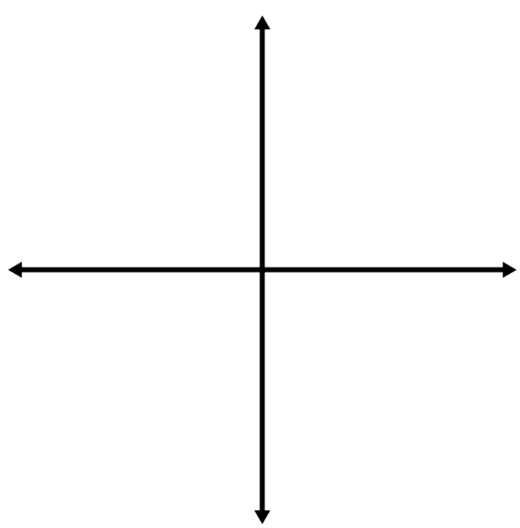
Graph $h(x) = \begin{cases} x^2 & \text{If } x < 4 \\ 4x - 9 & \text{If } x \geq 4 \end{cases}$

Graph $g(x) = \begin{cases} -1 & \text{If } x < -1 \\ x & \text{If } -1 \leq x \leq 1 \\ 1 & \text{If } x > 1 \end{cases}$

x	**H(x)**	**(x,y)**

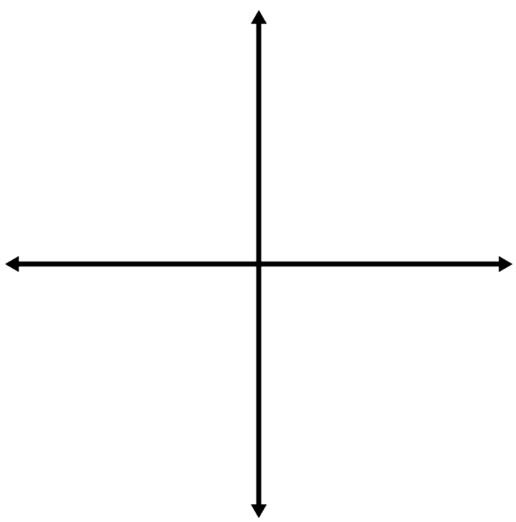
x	**G(x)**	**(x,y)**





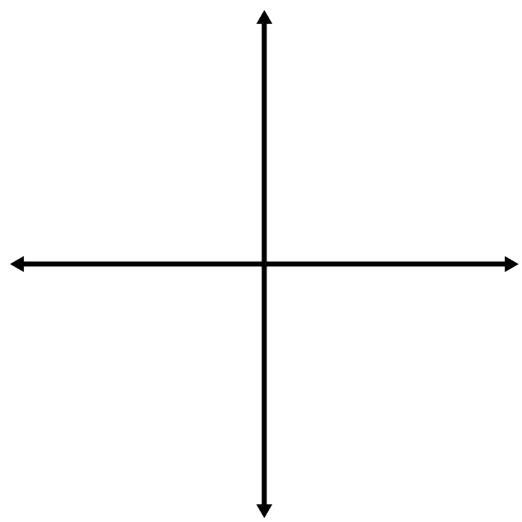
Graph $f(x) = (x-3)^2$

x	F(x)	(x,y)



Graph $g(x) = \begin{cases} 2x+3 & \text{If x} < -1 \\ 3-x & \text{If x} \geq -1 \end{cases}$

x	G(x)	(x,y)

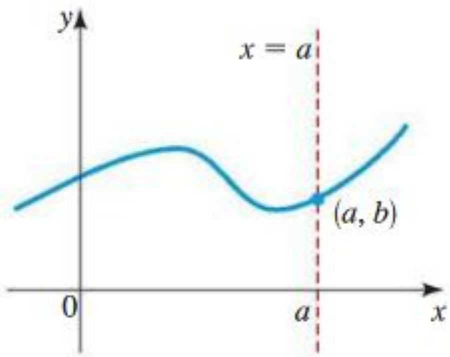


Vertical Line Test

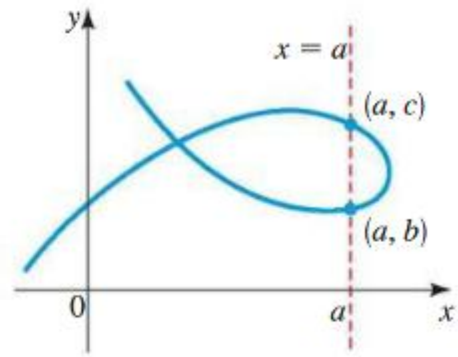
Not all curves are functions. For every given value of x that we plug into a function it returns a single value, called f(x). If this fact is violated then the curve is not a function. The following test allows us to see if a curve in the plane is a function or not.

> **THE VERTICAL LINE TEST**
>
> A curve in the coordinate plane is the graph of a function if and only if no vertical line intersects the curve more than once.

Examples of functions: Each of the graphs seen thus far in this section

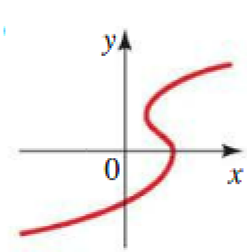


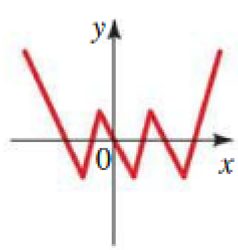
Graph of a function

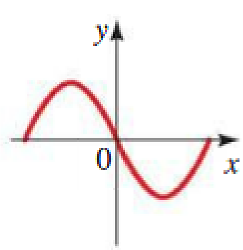


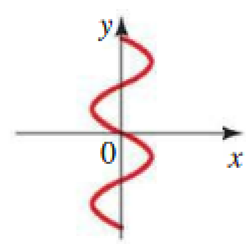
Not a graph of a function

Example: Use the vertical line test to determine which of the following curves are functions









2.3- Getting information From the Graph of a Function

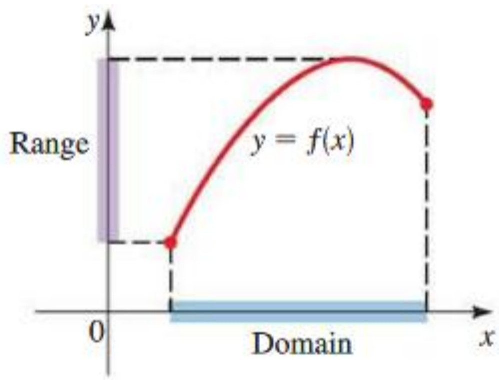
Looking at the graph of a function is one of the best and easiest ways to get information about the function being depicted.

Domain and Range

Range- can be thought of as the "height" of a graph. That is, how far the graph extends in both the positive, as well as negative, y-direction

Domain- can be thought of as the "width" of a graph. That is, how far the graph extends in both the positive, as well as negative, x-direction

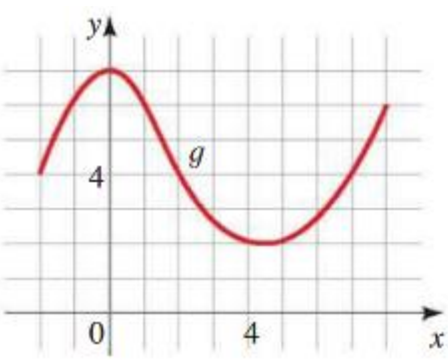
Reminder: we can write both the domain as well as the range is both set-builder notation as well as interval notation, keeping in mind the usual rules of "openness" and "closedness"

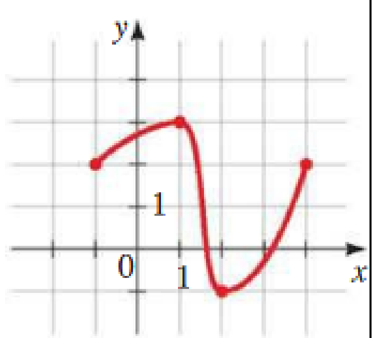


g(0)=

g(2)=

g(-2)=





f(1)=

f(2)=

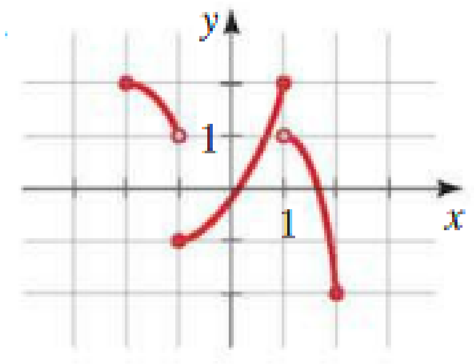
f(-3)=

Range:

Range:

Domain:

Domain:



Range:

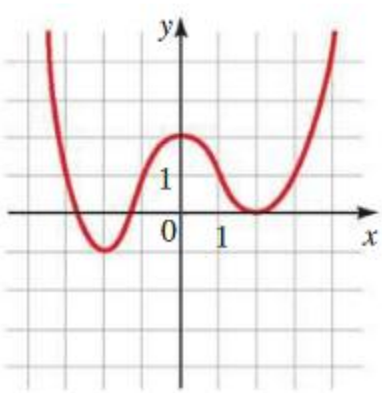
Domain:

f(-2)=

f(-1)=

f(1)=

(assume the graph continues in all directions)



Range:

Domain:

f(0)=

f(-2)=

f(4)=

2.5- Transformations of Functions

Vertical and Horizontal Shifting

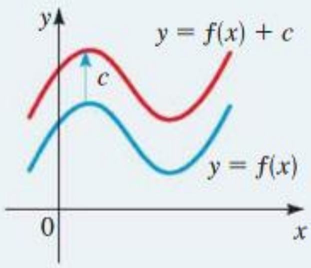
Drawn Exs:

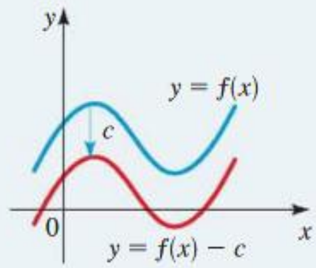
VERTICAL SHIFTS OF GRAPHS

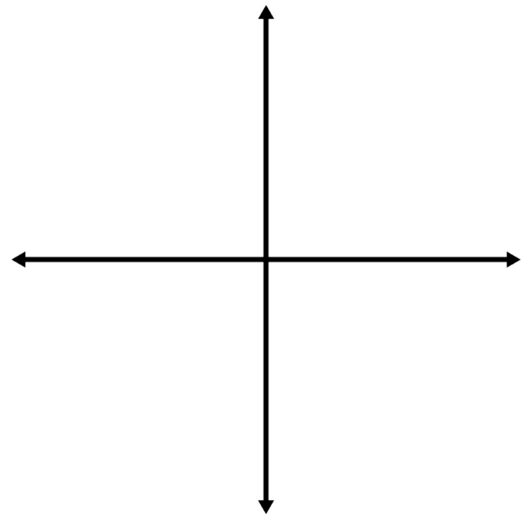
Suppose $c > 0$.

To graph $y = f(x) + c$, shift the graph of $y = f(x)$ upward c units.

To graph $y = f(x) - c$, shift the graph of $y = f(x)$ downward c units.





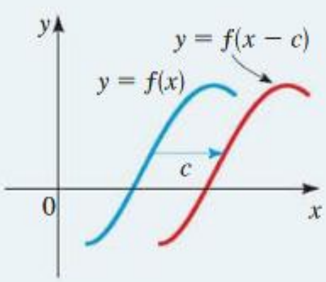


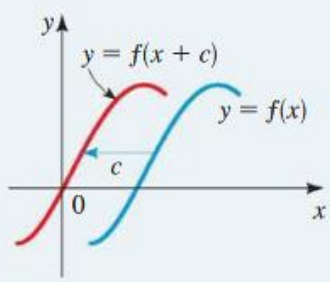
HORIZONTAL SHIFTS OF GRAPHS

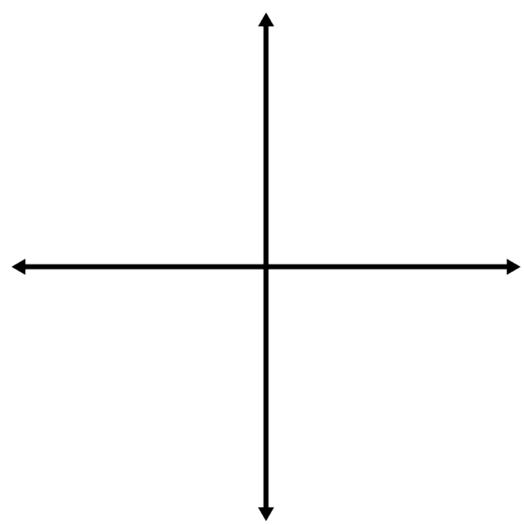
Suppose $c > 0$.

To graph $y = f(x - c)$, shift the graph of $y = f(x)$ to the right c units.

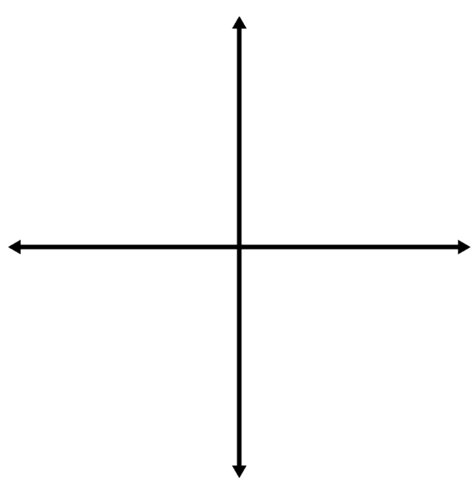
To graph $y = f(x + c)$, shift the graph of $y = f(x)$ to the left c units.







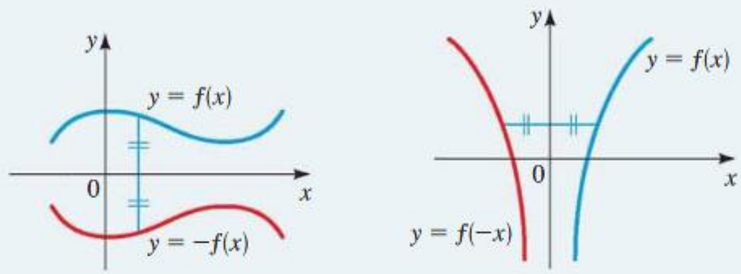
EXs using both:



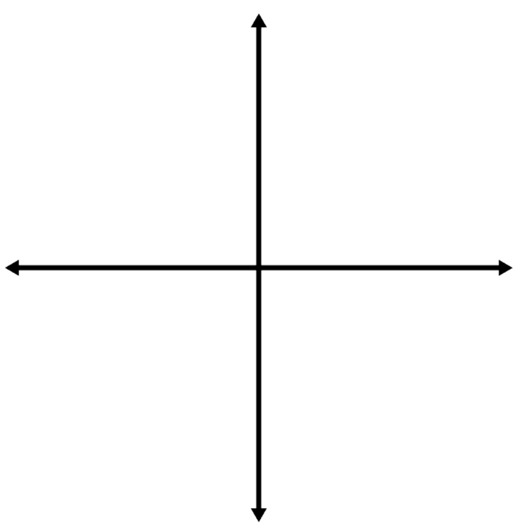
Reflecting Graphs

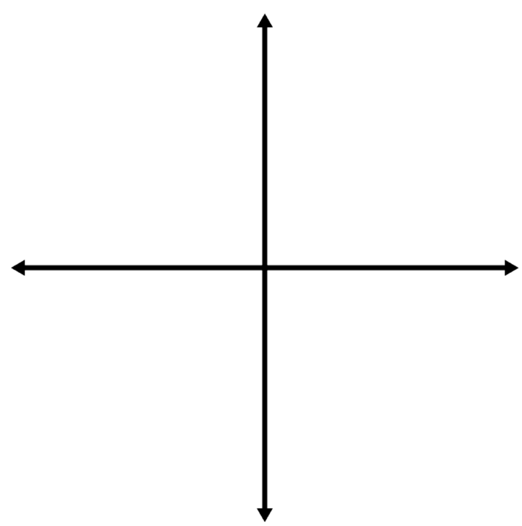
REFLECTING GRAPHS

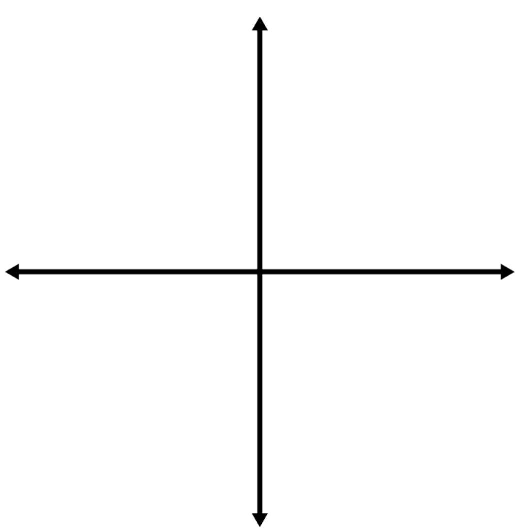
To graph $y = -f(x)$, reflect the graph of $y = f(x)$ in the x-axis.
To graph $y = f(-x)$, reflect the graph of $y = f(x)$ in the y-axis.

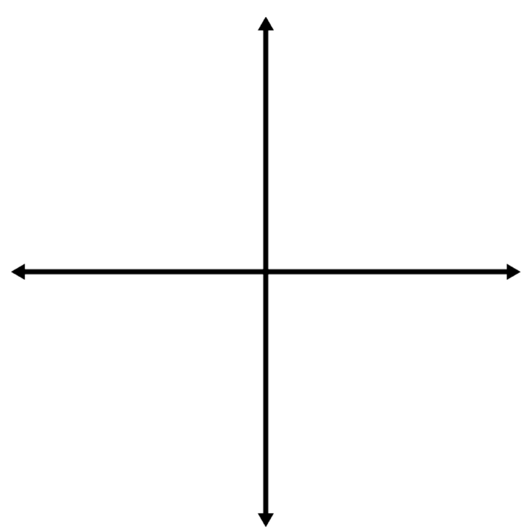


Drawn Exs:









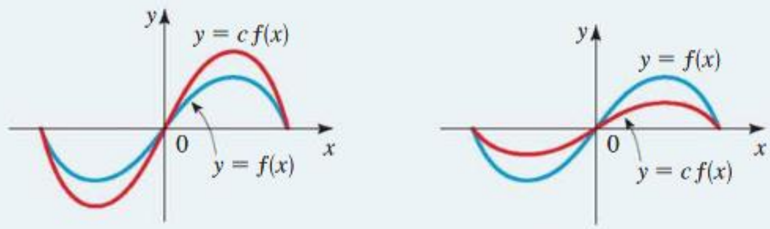
Vertical Stretching and Shrinking of Graphs

VERTICAL STRETCHING AND SHRINKING OF GRAPHS

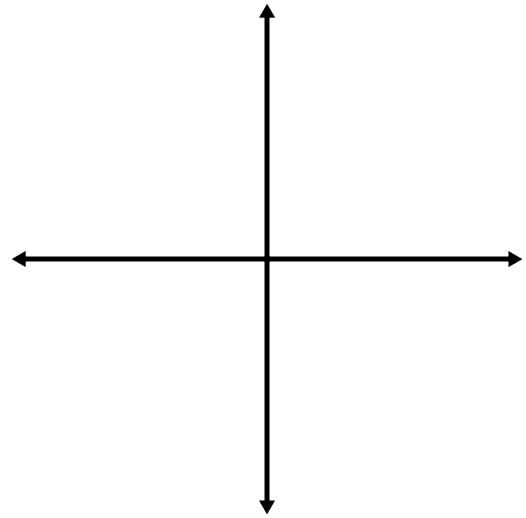
To graph $y = cf(x)$:

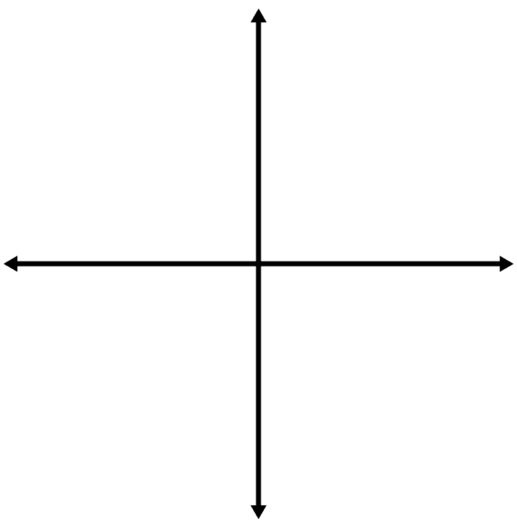
If $c > 1$, stretch the graph of $y = f(x)$ vertically by a factor of c.

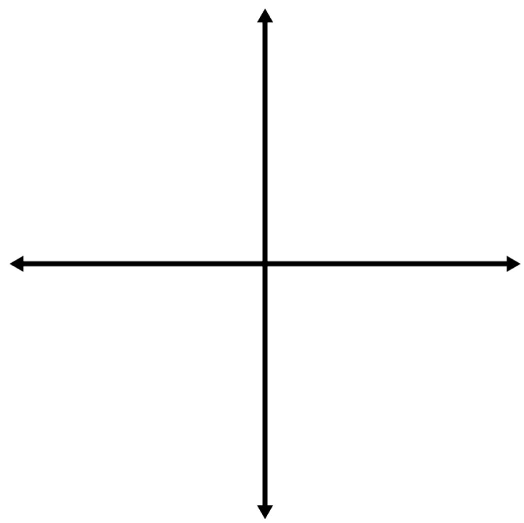
If $0 < c < 1$, shrink the graph of $y = f(x)$ vertically by a factor of c.

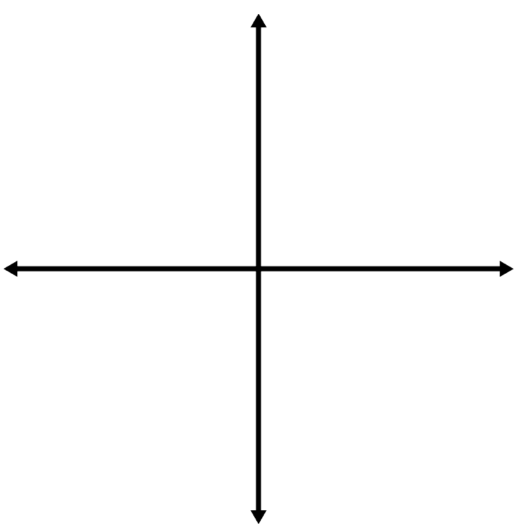


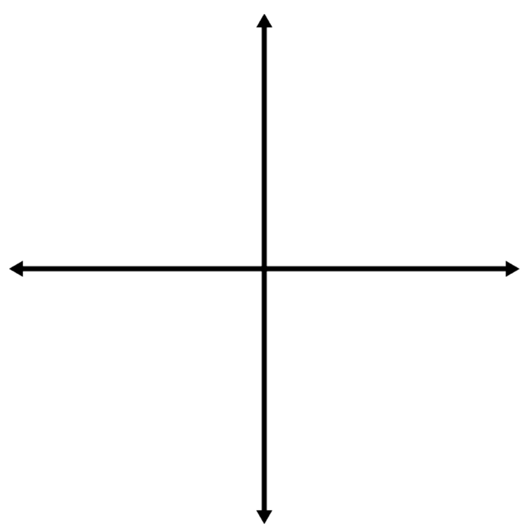
$c > 1$ $\qquad$ $0 < c < 1$











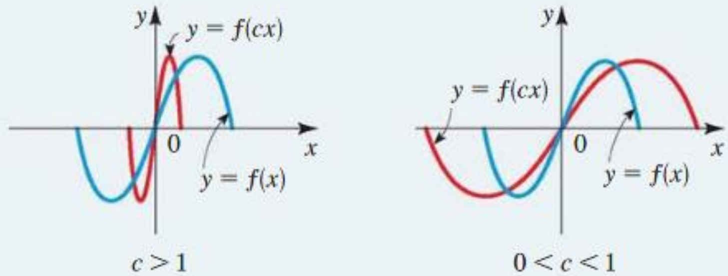
<u>Horizontal</u> Stretching and Shrinking

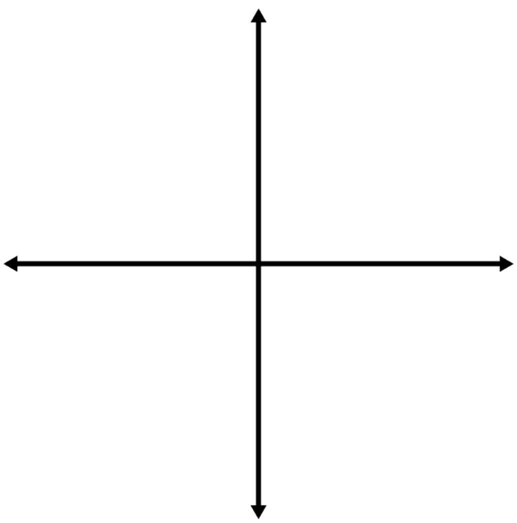
HORIZONTAL SHRINKING AND STRETCHING OF GRAPHS

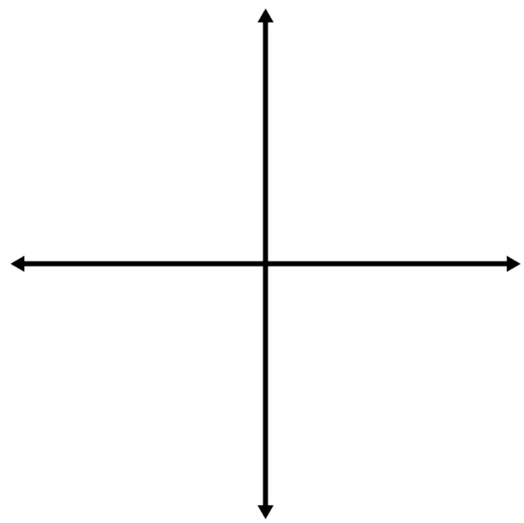
To graph $y = f(cx)$:

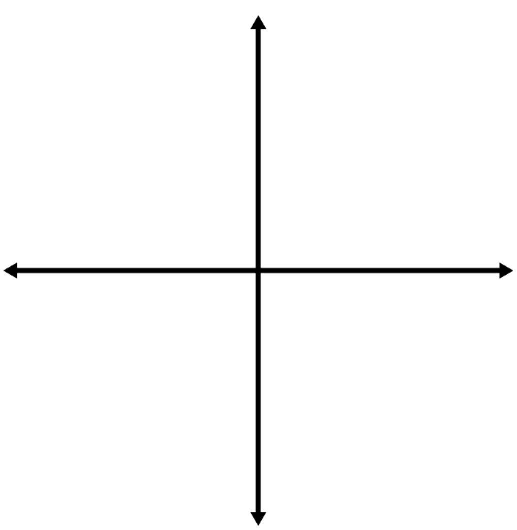
If $c > 1$, shrink the graph of $y = f(x)$ horizontally by a factor of $1/c$.

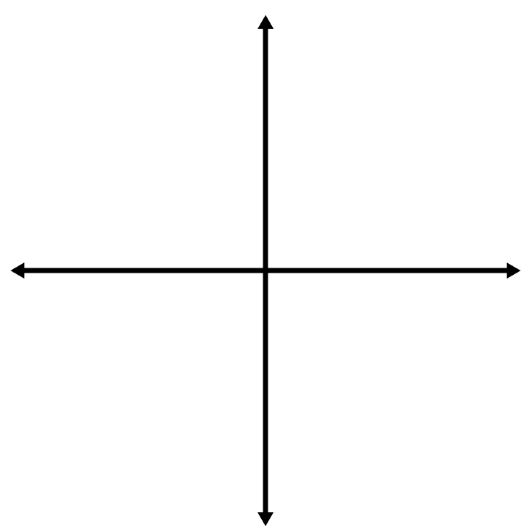
If $0 < c < 1$, stretch the graph of $y = f(x)$ horizontally by a factor of $1/c$.











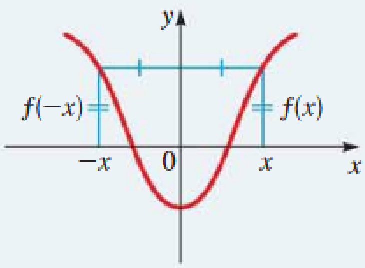
Even and Odd Functions

EVEN AND ODD FUNCTIONS

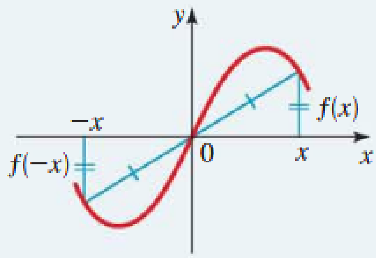
Let f be a function.

f is **even** if $f(-x) = f(x)$ for all x in the domain of f.

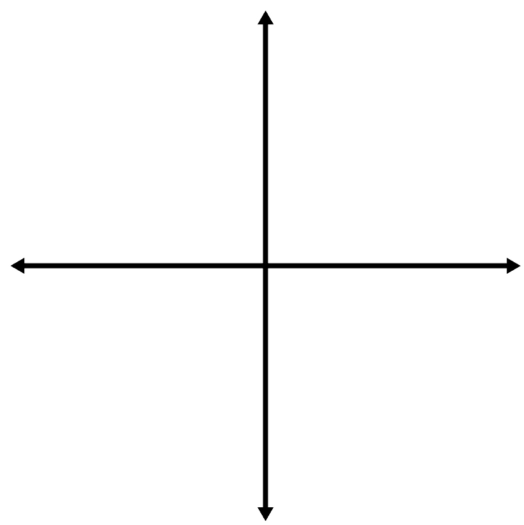
f is **odd** if $f(-x) = -f(x)$ for all x in the domain of f.

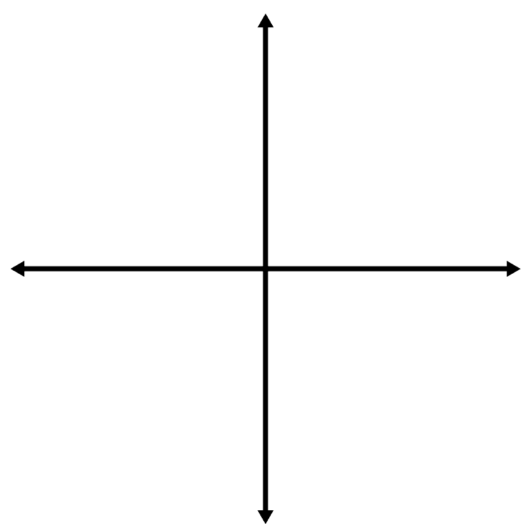


The graph of an even function is symmetric with respect to the y-axis.



The graph of an odd function is symmetric with respect to the origin.





2.6- Combining Functions

Sums, Differences, Products and Quotient

When we have two separate functions we can "combine" them using common operations such as addition, subtraction, multiplication and division.

ALGEBRA OF FUNCTIONS

Let f and g be functions with domains A and B. Then the functions $f+g$, $f-g$, fg, and f/g are defined as follows.

$(f+g)(x) = f(x) + g(x)$	Domain $A \cap B$
$(f-g)(x) = f(x) - g(x)$	Domain $A \cap B$
$(fg)(x) = f(x)g(x)$	Domain $A \cap B$
$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$	Domain $\{x \in A \cap B \mid g(x) \neq 0\}$

Domain- to determine the domain of the new combined function, write the domain of each individual function and then take the intersection of the two.

Ex: Let $f(x) = \frac{2}{x}$ and let $g(x) = \frac{4}{x+4}$

a) Find *(f+g)(x)*

Domain:

b) Find *(fg)(x)*

Domain:

c) Find $(\frac{f}{g})(x)$

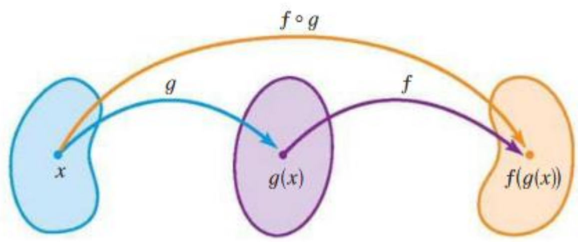
Domain:

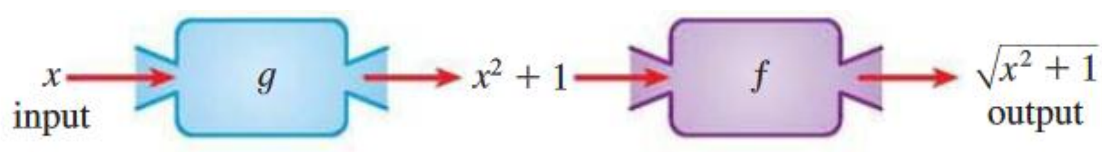
Composition of Functions

COMPOSITION OF FUNCTIONS

Given two functions f and g, the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$





Ex: Let $\boldsymbol{f(x) = x^3 + 2}$ and $\boldsymbol{g(x) = \sqrt[3]{x}}$

a) $(f \circ g)(x) =$ $\qquad$ $(f \circ g)(2) =$

b) $(g \circ f)(x) =$ $\qquad$ $(g \circ f)(3) =$

c) $(f \circ f)(x) =$ $\qquad$ $(g \circ g)(x) =$

We can have a composition of 3 or more functions as well. To do this simply take the composition one pieces at a time:

Ex: Let $\boldsymbol{f(y) = y - 1}$, $\boldsymbol{g(y) = \sqrt{y}}$ **and** $\boldsymbol{h(y) = y + 1}$

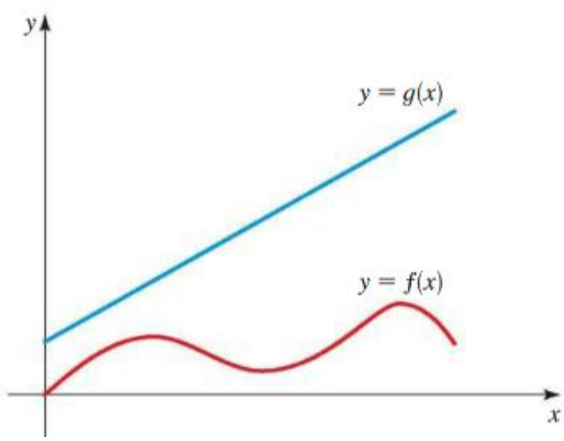
Find $$(f \circ g \circ h)(y) = f(g(h(y)))$$

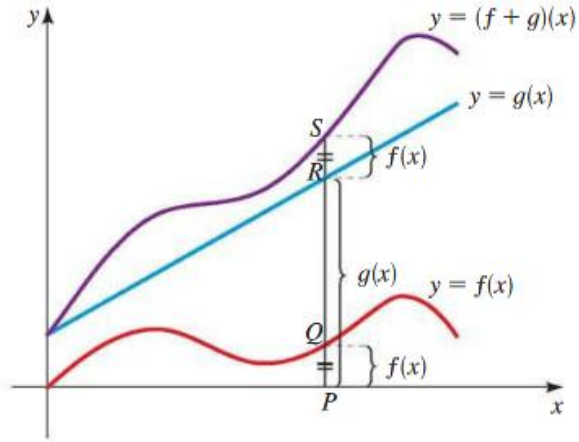
<u>Recognizing a Composition of Functions</u>

When given a more complicated looking function, we can often decompose it into simpler ones. To do this, consider all types of functions involved in the composition function.

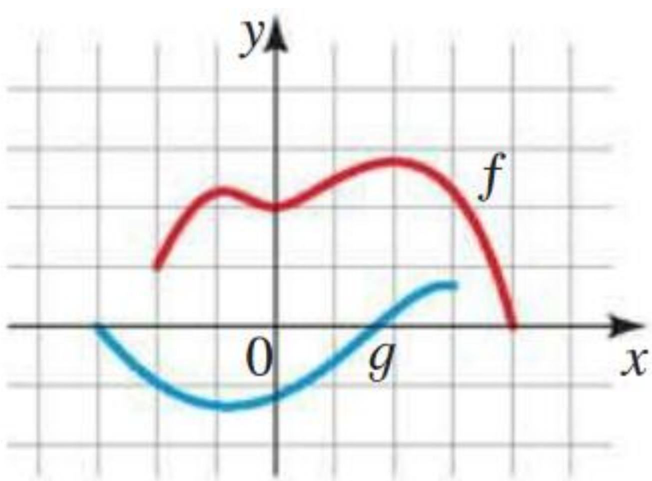
$\sqrt{x} + 1$ $\qquad\qquad$ $\frac{1}{x+3}$ $\qquad\qquad$ $\sqrt{1 + \sqrt{x}}$

Using Graphical Addition





Use graphical addition to sketch the graph of $f + g$



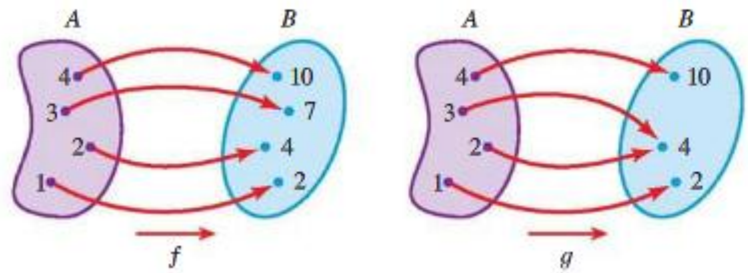
<u>2.7- One-to-One Functions and Their Inverses</u>

One-to-One Functions

DEFINITION OF A ONE-TO-ONE FUNCTION

A function with domain A is called a **one-to-one function** if no two elements of A have the same image, that is,

$$f(x_1) \neq f(x_2) \quad \text{whenever } x_1 \neq x_2$$



f is one-to-one $\qquad$ g is not one-to-one

Two ways to determine if a function is one-to-one:

a) Logically
b) Graphically

a) <u>Logically</u>: If a function sends two separate input values (x-values) to the same output value (y-value) then the function is <u>not</u> one-to-one. This method is more difficult because you have to consider all possible values in the domain and think about which may be sent to the same value.

Ex: Is x^2 one-to-one?

Ex: Is $x^2 + y^2 = 10$ one-to-one?

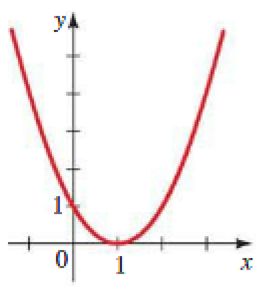
Ex: Is x^3 one-to-one?

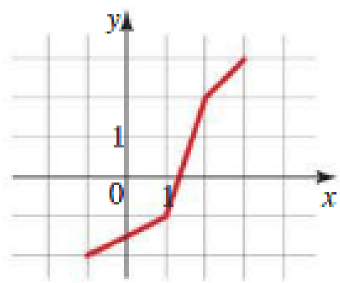
b) <u>Graphically</u>: If a function is one-to-one then each output value (y-value) is achieved by only a <u>single</u> input value (x-value). If you are looking at the graph of the function this means that for every y-value there is only one corresponding x-value

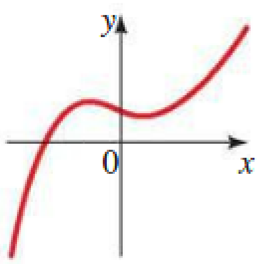
> **HORIZONTAL LINE TEST**
>
> A function is one-to-one if and only if no horizontal line intersects its graph more than once.

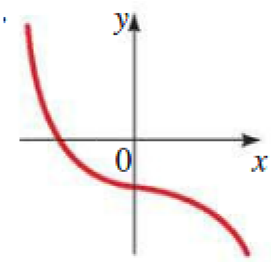
<u>Examples</u>

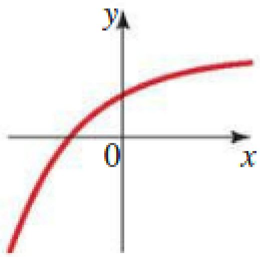
$g(x) = (x - 1)^2$

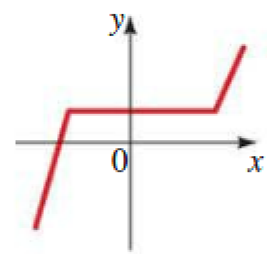












The Inverse of a Function

When we think about functions usually we think about plugging in an x-value and getting out a y-value. However, the inverse of a one-to-one function does the opposite. Given a y-value of a one-to-one function, the inverse tells you the corresponding x-value from which it came.

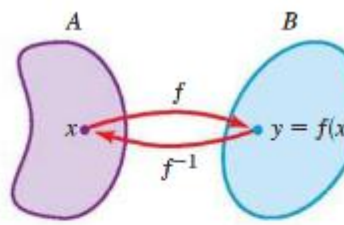
DEFINITION OF THE INVERSE OF A FUNCTION

Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$$

for any y in B.

<u>Notice:</u> Only one-to-one functions have inverses

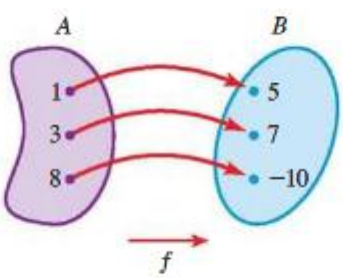


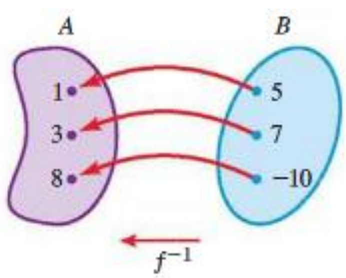
<u>Notation:</u>

If $f(x) = y$ then $f^{-1}(y) = x$.

$$\boldsymbol{f^{-1}(y) \neq \frac{1}{f(y)}}$$

$f(1) = 5$ $f(3) = 7$ $f(8) = -10$	$f^{-1}(5) = 1$ $f^{-1}(7) = 3$ $f^{-1}(-10) = 8$





Ex: If $f(26) = 109$ then $f^{-1}(109) =$

…Inverses continued…

INVERSE FUNCTION PROPERTY

Let f be a one-to-one function with domain A and range B. The inverse function f^{-1} satisfies the following cancellation properties:

$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in } A$$

$$f(f^{-1}(x)) = x \quad \text{for every } x \text{ in } B$$

Conversely, any function f^{-1} satisfying these equations is the inverse of f.

This provides a good way to check if two functions are inverses of each other- see examples below

Ex: Let $f(x) = x^{\frac{2}{5}}$ and let $g(x) = x^{\frac{5}{2}}$. Verify that the two functions are each other's inverse.

HOW TO FIND THE INVERSE OF A ONE-TO-ONE FUNCTION

1. Write $y = f(x)$.
2. Solve this equation for x in terms of y (if possible).
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.

Ex: Find the inverse of the following function and verify that it is indeed the inverse using the inverse property function

$$f(x) = 5x + 10$$

…Inverses continued…

If the function is a rational function then you will have to first multiply both sides of the equation by the denominator, then carry out any distribution and finally isolate the x.

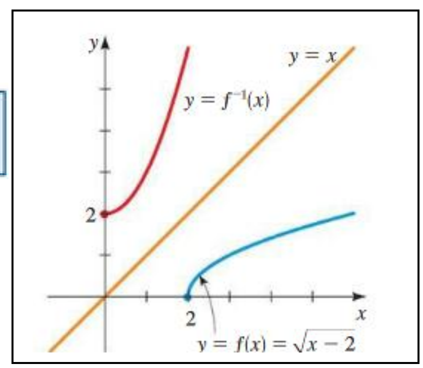
Note: you may need to do some factoring in order to isolate the x variable

Ex: Find the inverse of the following function

$$f(x) = \frac{x-2}{x+2}$$

Graphing the Inverse of a Function

The graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$.



3.2- Polynomial Functions and Their Graphs

Some Terminology:

POLYNOMIAL FUNCTIONS

A **polynomial function of degree n** is a function of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

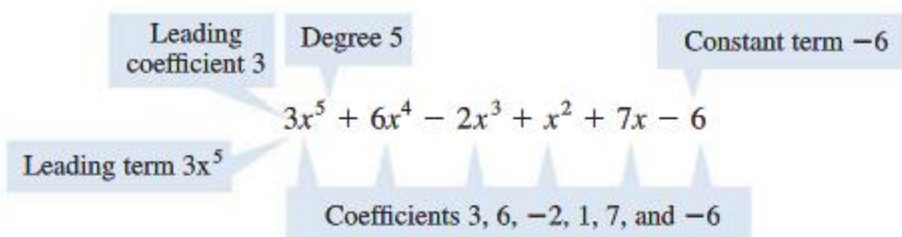
where n is a nonnegative integer and $a_n \neq 0$.

The numbers $a_0, a_1, a_2, \ldots, a_n$ are called the **coefficients** of the polynomial.

The number a_0 is the **constant coefficient** or **constant term**.

The number a_n, the coefficient of the highest power, is the **leading coefficient**, and the term $a_n x^n$ is the **leading term**.

Ex :



$P(x) = 3$	Degree 0
$Q(x) = 4x - 7$	Degree 1
$R(x) = x^2 + x$	Degree 2
$S(x) = 2x^3 - 6x^2 - 10$	Degree 3

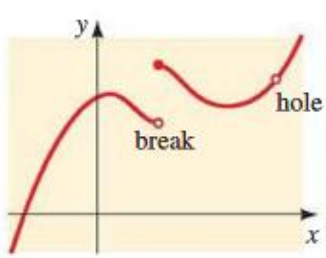
Polynomials of degree 0 and 1 are straight lines.

Polynomials of degree 2 are parabolas.

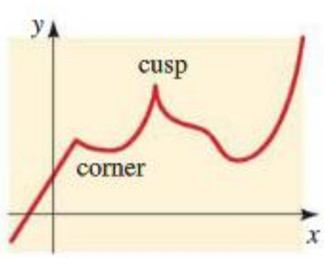
The higher the degree, the more complicated the graph.

Graphs of polynomials must be **continuous,** which means the graphs cannot have any breaks or holes. (see examples below)

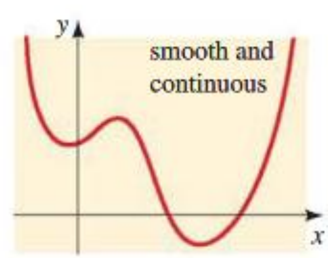
Exs:



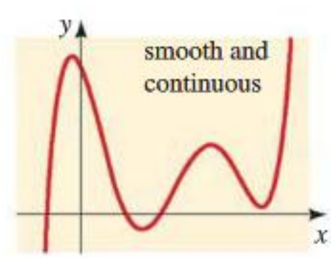
Not the graph of a polynomial function



Not the graph of a polynomial function



Graph of a polynomial function

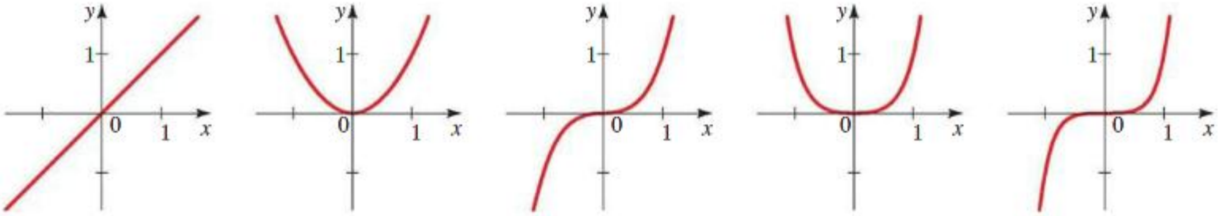


Graph of a polynomial function

<u>Graphing Basic Polynomials</u>

<u>Monomial-</u> a polynomial of the form x^n where *n* is a natural number

Some simple monomials you should know the shape of:



(a) $y = x$ (b) $y = x^2$ (c) $y = x^3$ (d) $y = x^4$ (e) $y = x^5$

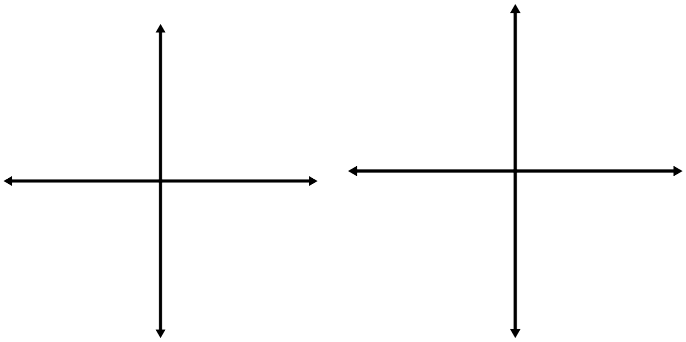
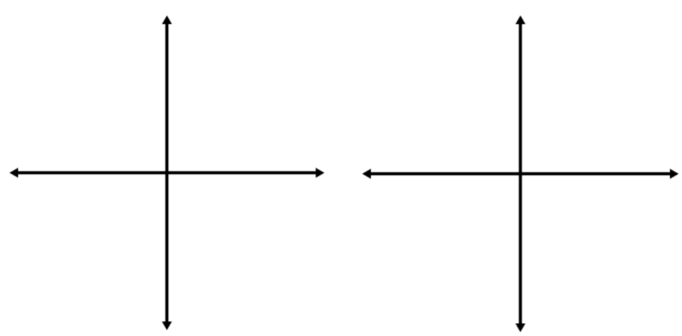
<u>Two ways to graph a polynomial:</u>

1) **Use information from section 2.5 about transformation a graph and pair it with information about the shape of a known monomial.**
    a. This method only works for more “simple” polynomials

2) **Use the guidelines for graphing polynomial functions**
    a. This method works for any polynomial but requires more work

1) If a function is simple enough to understand the shape of its graph then use information about shifting, shrinking, stretching and flipping to graph the function. This information can be found in section 2.5.

Exs:

Graph $(x-4)^2$	Graph $-2(x-2)^5$
	

…Graphing Polynomials Continued…

2) If a polynomial is too complicated to draw using method #1 then use the following information to graph it

GUIDELINES FOR GRAPHING POLYNOMIAL FUNCTIONS

1. **Zeros.** Factor the polynomial to find all its real zeros; these are the x-intercepts of the graph.
2. **Test Points.** Make a table of values for the polynomial. Include test points to determine whether the graph of the polynomial lies above or below the x-axis on the intervals determined by the zeros. Include the y-intercept in the table.
3. **End Behavior.** Determine the end behavior of the polynomial.
4. **Graph.** Plot the intercepts and other points you found in the table. Sketch a smooth curve that passes through these points and exhibits the required end behavior.

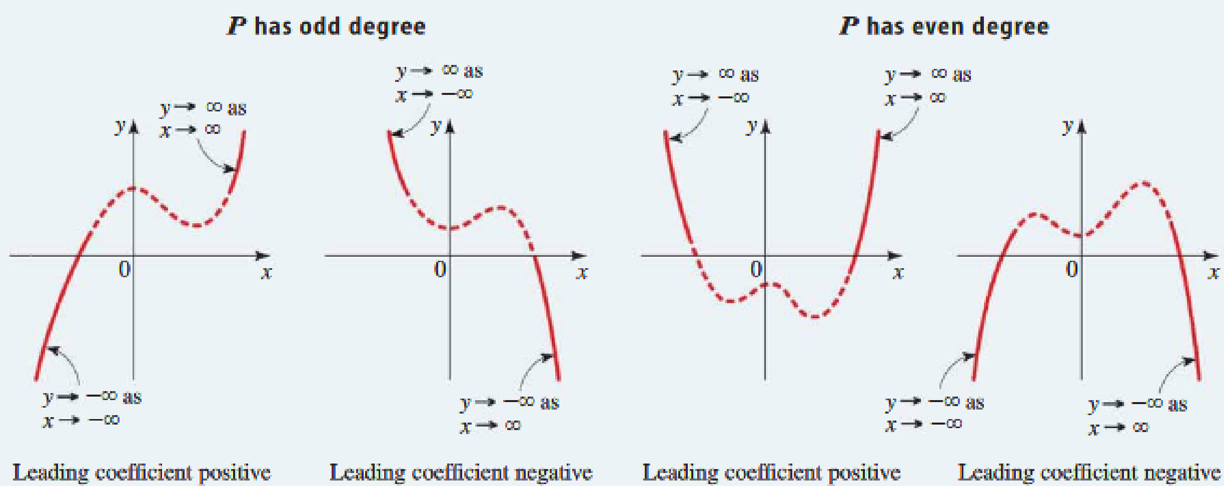
Step 1: Factor the polynomials and write down all x-values that result in f(x)=0

Step 2: Make a table using values found in step 1. Plug a single value into the function that can be found between each test value to decide if the graph is positive or negative at those points.

Step 3: Determine the end behavior using the following information:

END BEHAVIOR OF POLYNOMIALS

The end behavior of the polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is determined by the degree n and the sign of the leading coefficient a_n, as indicated in the following graphs.



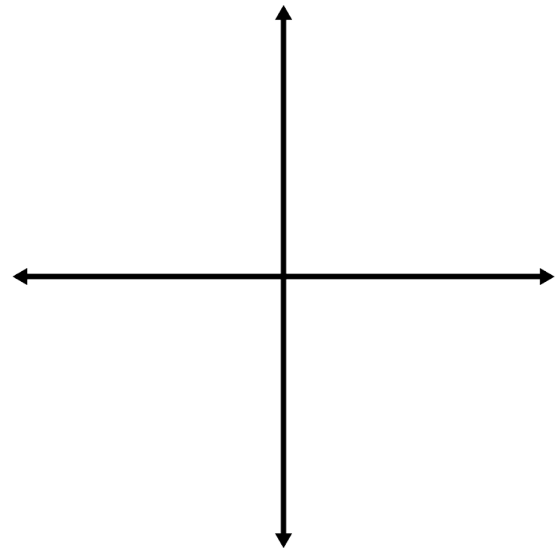
Step 4: Plot each x-intercept found in step 1 on the x-y axis. Use this information paired with the information from step 2 to connect each x-intercept. Use the information found in step 3 to determine the end behavior of the graph.

See next page for some examples

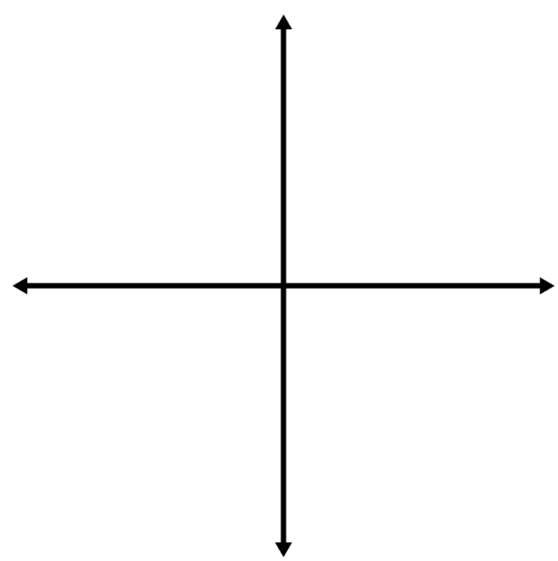
Examples of graphing polynomials

Ex: Use steps 1-4 from the previous page to graph the following polynomials:

$$g(x) = x^3 + 2x^2 - 8x$$



$$f(x) = x^4 - 3x^3 + 2x^2$$



<u>3.3- Dividing Polynomials</u>

Remember how to do long division?

Divide 227 by 3 $\qquad\qquad$ $107 \div 5$

We can perform long division on polynomials in a similar manor.

<u>Steps for performing long division on a polynomial:</u>

1) Set up the problem with the polynomial doing the dividing (<u>called divisor</u>) outside the division operator and the polynomial being divided (<u>called dividend</u>) under the division operator. <u>Remember to write any missing variable exponents as a 0</u>.
2) Just like in the examples above, you perform long division one term at a time. First, think about what you would need to multiply the first term of the divisor by that would result in the first term of the dividend and write your answer above the first term of the dividend.
3) Multiply your answer from 2 through your divisor using the distributive property and write your answer below the dividend.
4) Subtract your answer to 3 from the dividend
5) Repeat steps 2-4 for each subsequent polynomial until you cannot perform any more operations. That is, a reminder will result when the degree becomes less than the degree of the divisor.
6) Account for a remainder

Ex: $\frac{x^3-x^2-2x+6}{x-2}$

DIVISION ALGORITHM

If $P(x)$ and $D(x)$ are polynomials, with $D(x) \neq 0$, then there exist unique polynomials $Q(x)$ and $R(x)$, where $R(x)$ is either 0 or of degree less than the degree of $D(x)$, such that

$$\underbrace{P(x)}_{\text{Dividend}} = \underbrace{D(x)}_{\text{Divisor}} \cdot \underbrace{Q(x)}_{\text{Quotient}} + \underbrace{R(x)}_{\text{Remainder}}$$

The polynomials $P(x)$ and $D(x)$ are called the **dividend** and **divisor**, respectively, $Q(x)$ is the **quotient**, and $R(x)$ is the **remainder**.

$$\frac{3x^4-5x^3-20x-5}{x^2+x+3}$$

....**long division continued…**

FACTOR THEOREM

c is a zero of P if and only if $x - c$ is a factor of $P(x)$.

This means that if *c* is a factor then *x-c* should evenly divide the polynomial.

This means there should be no remainder when you divide the polynomial by *x-c.*

Ex: Use the factor theorem to show that $x - 2$ is a factor of $P(x) = x^3 + 2x^2 - 3x - 10$

If a polynomial $P(x)$has zeros $c_1, c_2 \dots c_n$ of, then $P(x)$ can be factored into the product

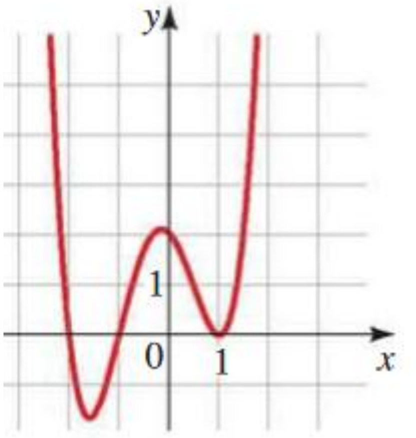
$$P(x) = (x - c_1)(x - c_2) \dots (x - c_n)$$

This means that the polynomial can be found by completing the multiplication shown above

Ex: Suppose $P(x)$has zeros at $2, 5$ and -6. Find $P(x)$.

Find the polynomial of the specified degree whose graph is shown

Degree 4



<u>3.7-Rational Functions</u>

<u>Rational Function:</u> A function of the form $\frac{P(x)}{Q(x)}$ where both P and Q are polynomials

The <u>Domain</u> of a rational function is all x-values excluding those that make the denominator zero

The <u>Range</u> of a rational function is all possible y-values that can result from plugging in any x-value in the domain

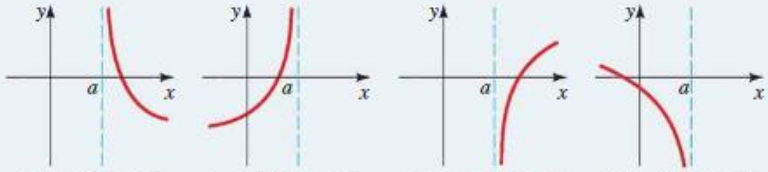
Ex: Find the domain and range of $f(x) = \frac{1}{x}$

Test some values as x$\rightarrow 0$ from both directions and x $\rightarrow \pm\infty$

The tables above provide an introduction to the idea of asymptotes…

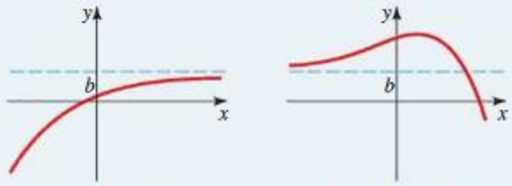
DEFINITION OF VERTICAL AND HORIZONTAL ASYMPTOTES

1. The line $x = a$ is a **vertical asymptote** of the function $y = f(x)$ if y approaches $\pm\infty$ as x approaches a from the right or left.



$y \to \infty$ as $x \to a^+$ $\quad$ $y \to \infty$ as $x \to a^-$ $\quad$ $y \to -\infty$ as $x \to a^+$ $\quad$ $y \to -\infty$ as $x \to a^-$

2. The line $y = b$ is a **horizontal asymptote** of the function $y = f(x)$ if y approaches b as x approaches $\pm\infty$.



$y \to b$ as $x \to \infty$ $\quad$ $y \to b$ as $x \to -\infty$

<u>Vertical Asymptote(s)</u> of a rational function occur at the x-value(s) that make the denominator zero.

<u>Horizontal Asymptote(s)</u> of a rational function occur at the y-value(s) that the function approach as $x \to \infty$ and/or as $x \to -\infty$.

FINDING ASYMPTOTES OF RATIONAL FUNCTIONS

Let r be the rational function

$$r(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

1. The vertical asymptotes of r are the lines $x = a$, where a is a zero of the denominator.
2. **(a)** If $n < m$, then r has horizontal asymptote $y = 0$.

   (b) If $n = m$, then r has horizontal asymptote $y = \frac{a_n}{b_m}$.

   (c) If $n > m$, then r has no horizontal asymptote.

$$f(x) = \frac{2x-3}{x^2-1} =$$

<u>Vertical Asymptote(s)</u> $\qquad\qquad\qquad\qquad$ <u>Horizontal Asymptote(s)</u>

$$g(x) = \frac{3x^2 - 2}{1 - 2x^2}$$

<u>Vertical Asymptote(s)</u>

<u>Horizontal Asymptote(s)</u>

$$f(x) = \frac{x^3}{x^2 + 5}$$

<u>Vertical Asymptote(s)</u>

<u>Horizontal Asymptote(s)</u>

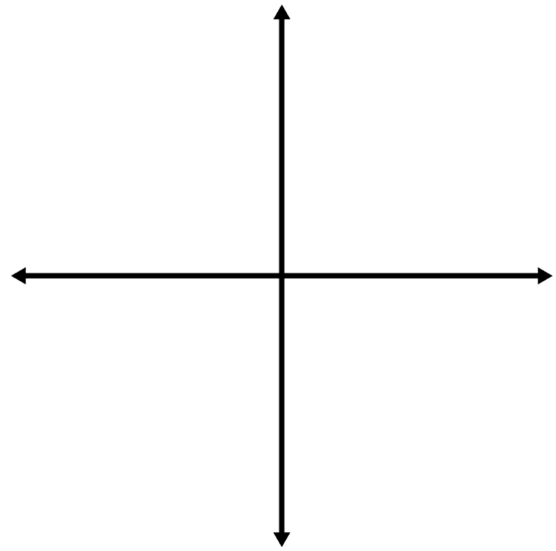
<u>Graphing Rational Functions</u>

SKETCHING GRAPHS OF RATIONAL FUNCTIONS

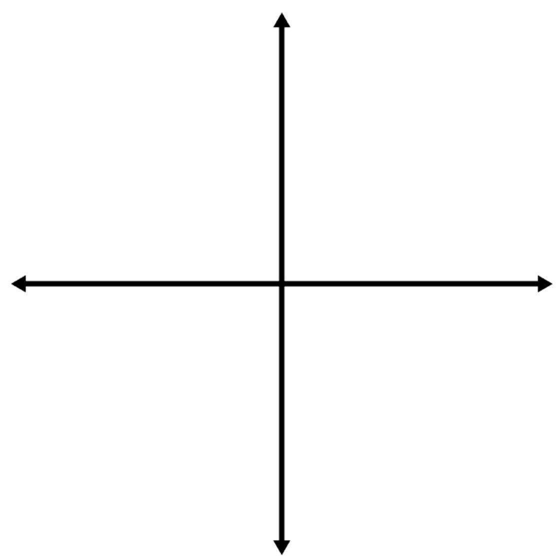
1. **Factor.** Factor the numerator and denominator.
2. **Intercepts.** Find the x-intercepts by determining the zeros of the numerator and the y-intercept from the value of the function at $x = 0$.
3. **Vertical Asymptotes.** Find the vertical asymptotes by determining the zeros of the denominator, and then see whether $y \to \infty$ or $y \to -\infty$ on each side of each vertical asymptote by using test values.
4. **Horizontal Asymptote.** Find the horizontal asymptote (if any), using the procedure described in the box on page 282.
5. **Sketch the Graph.** Graph the information provided by the first four steps. Then plot as many additional points as needed to fill in the rest of the graph of the function.

<u>See next page for examples</u>

Graph $\frac{x-2}{(\mathrm{x}+1)^2}$



Graph $\frac{4x^2}{\mathrm{x}^2-2x-3}$



4.1 & 4.2- Exponential Functions

EXPONENTIAL FUNCTIONS

The **exponential function with base *a*** is defined for all real numbers x by

$$f(x) = a^x$$

where $a > 0$ and $a \neq 1$.

$$f(x) = 2^x \qquad g(x) = 3^x \qquad h(x) = 10^x$$

Base 2   Base 3   Base 10

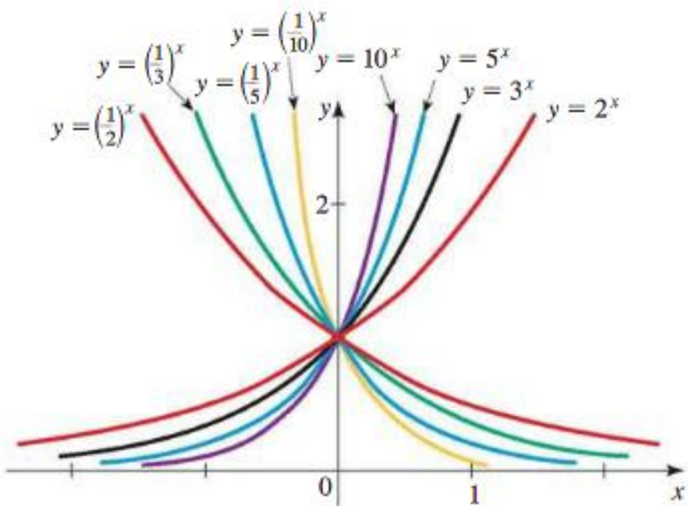


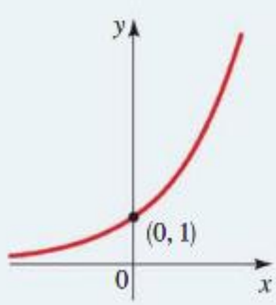
Figure 1

GRAPHS OF EXPONENTIAL FUNCTIONS

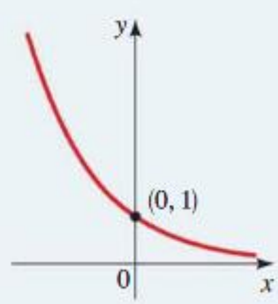
The exponential function

$$f(x) = a^x \qquad (a > 0, a \neq 1)$$

has domain $\mathbb{R}$ and range $(0, \infty)$. The line $y = 0$ (the x-axis) is a horizontal asymptote of f. The graph of f has one of the following shapes.



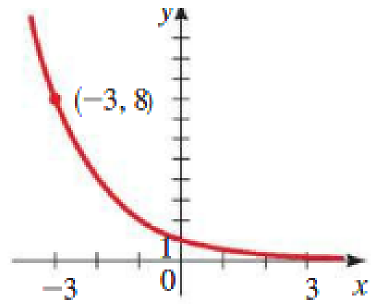
$f(x) = a^x$ for $a > 1$



$f(x) = a^x$ for $0 < a < 1$

Ex: Let $f(x) = 3^x$ and find $f(2)$, $f(\frac{3}{2})$

If you know your function is of the form $y = a^x$ and you know a single point that your function passes through you can figure out what your base, a, is. Just plug in the point, use information from the previous page and solve the equation.

Suppose $y = a^x$ for some base a and that the function passes through the point (4 , 16). Find *a*	Suppose $y = a^x$ for some base a. Determine what the base is based on the following graph. 

THE NATURAL EXPONENTIAL FUNCTION

The **natural exponential function** is the exponential function

$$f(x) = e^x$$

with base e. It is often referred to as *the* exponential function.

Where $e \approx 2.71828182845904523536$...

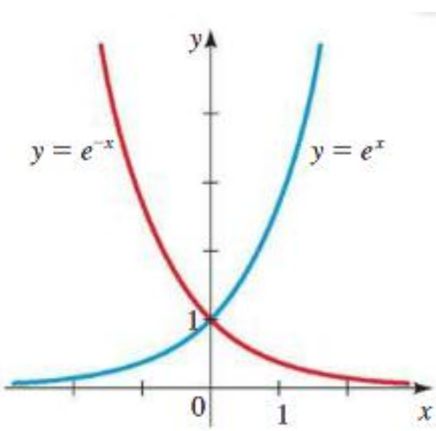
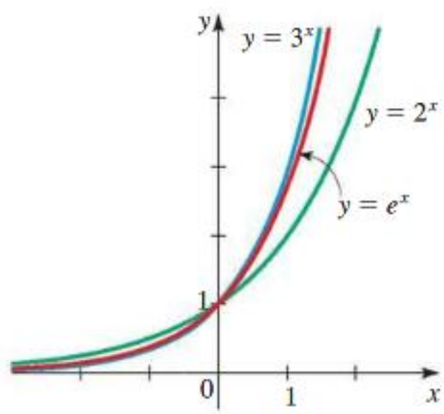


Figure 2



Graphing Exponential Functions:

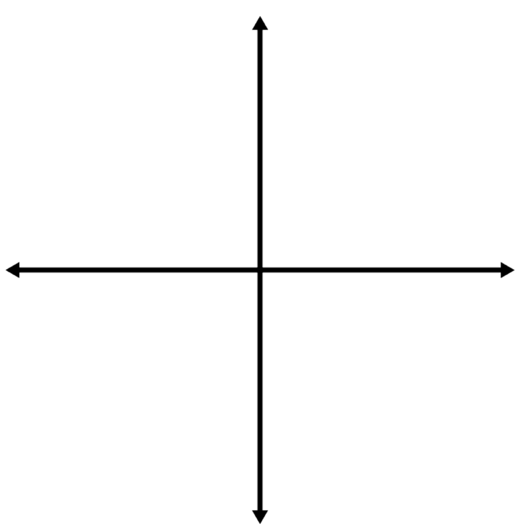
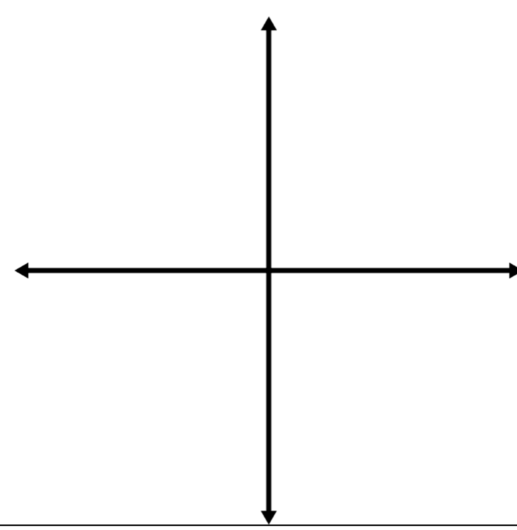
Using the above information and pictures, before graphing an exponential you should already have a good idea of what it looks like. There are two methods for fine-tuning your drawing:

A) Using a table (used only for functions with some base $a \neq e$)
B) Using transformation properties from section 2.5

Graphing Using a table

1) Using the information on previous pages, understand the general shape before you even start
   a. Is it increasing or decreasing?
2) Create a table of values
   a. Make sure you include x=0 so you can determine the y-intercept
   b. Make sure you choose enough values so that you understand how the function behaves in both the positive x-direction as well as the negative x-direction
3) Plot the values found in step 2 and connect the dots

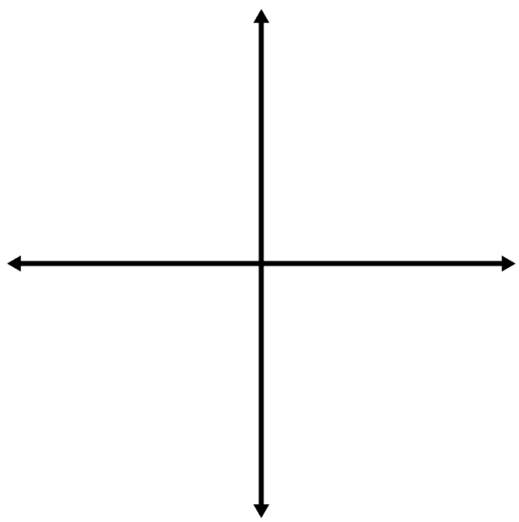
Exs:

Plot $g(x) = (2)^{-x}$	Plot $h(x) = 2(\frac{1}{4})^x$
	

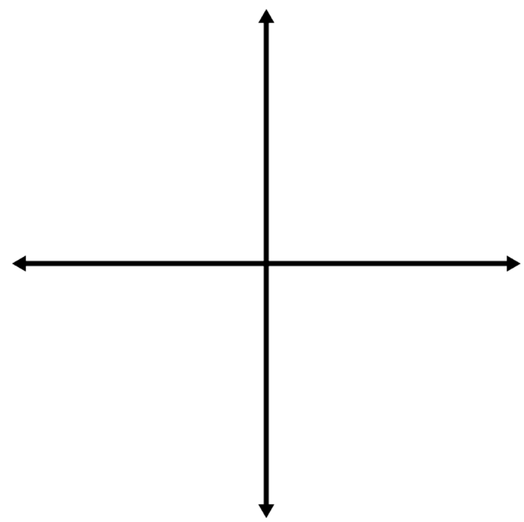
Graphing Using Transformations from section 2.5

1) Using the information on previous pages, understand the general shape before you even start
    a. Look at figure 1 and figure 2 to determine the shape
    b. Is it increasing or decreasing?
2) Plug in x=0 to determine the y-intercept and plot it.
3) Shift the graph appropriately using the information in section 2.5

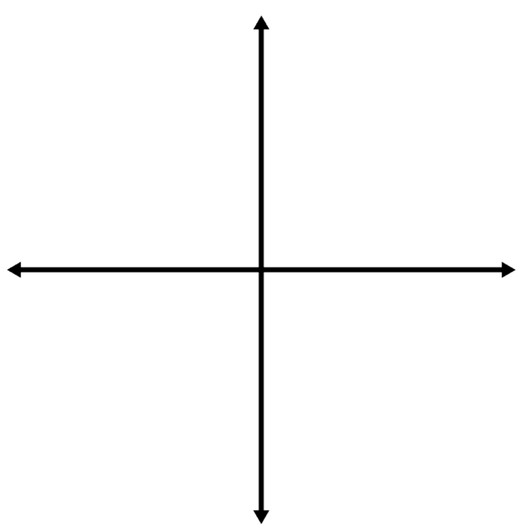
Plot $g(x) = (10)^{-x}$



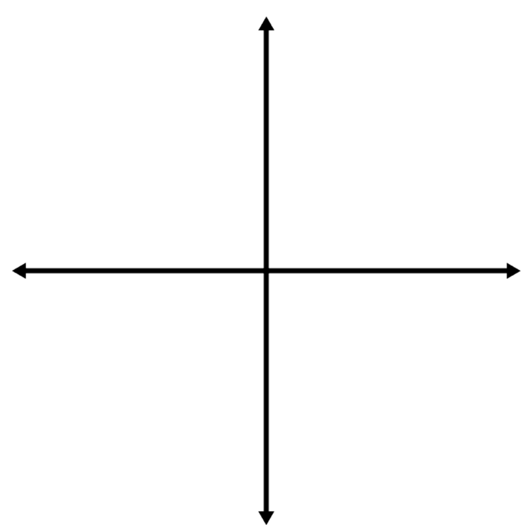
Plot $f(x) = (2)^{x-4} + 1$



Plot $h(x) = 1 + e^{x}$

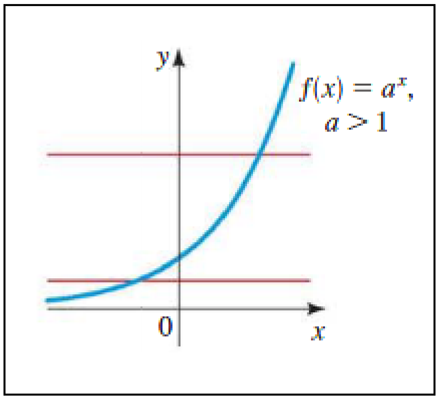


Plot $g(x) = -e^{x-1} - 2$



4.3- Logarithmic Functions

Notice that, by the horizontal line test, every exponential function is one-to-one (see picture below). From section 2.7, this means that every exponential function has an inverse.



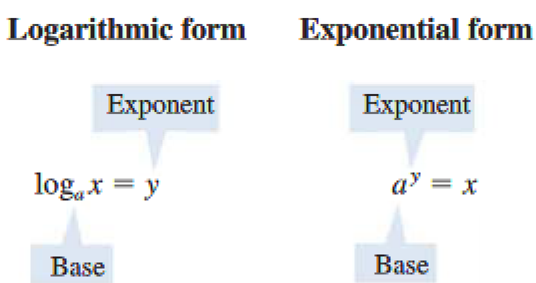
That is, for every function of the form $f(x) = a^x$, there exists an inverse function such that $f^{-1}(y) = x$

DEFINITION OF THE LOGARITHMIC FUNCTION

Let a be a positive number with $a \neq 1$. The **logarithmic function with base a**, denoted by $\log_a$, is defined by

$$\log_a x = y \quad \Leftrightarrow \quad a^y = x$$

So $\log_a x$ is the *exponent* to which the base a must be raised to give x.



When working with logarithms you must keep in mind their equivalent exponential forms shown above and to the left. The exponential form is easier to work with and because whatever is true for the exponential form must also be true fo the logarithmic form, it makes working out the problems much easier.

Evaluating logarithms- first change the logarithm to an exponential and then evaluate

Keep in mind the following properties:

PROPERTIES OF LOGARITHMS

Property	Reason
1. $\log_a 1 = 0$	We must raise a to the power 0 to get 1.
2. $\log_a a = 1$	We must raise a to the power 1 to get a.
3. $\log_a a^x = x$	We must raise a to the power x to get a^x.
4. $a^{\log_a x} = x$	$\log_a x$ is the power to which a must be raised to get x.

$8^{-1} = \frac{1}{8}$	$10^3 = 1000$	$7^3 = 343$
$\log_5 5^4$	$\log_{10} \sqrt{10}$	$\log_4 \frac{1}{2}$

Common and Natural Logarithms

COMMON LOGARITHM

The logarithm with base 10 is called the **common logarithm** and is denoted by omitting the base:

$$\log x = \log_{10} x$$

NATURAL LOGARITHM

The logarithm with base e is called the **natural logarithm** and is denoted by **ln**:

$$\ln x = \log_e x$$

$$\ln x = y \quad \Leftrightarrow \quad e^y = x$$

PROPERTIES OF NATURAL LOGARITHMS

Property	Reason
1. $\ln 1 = 0$	We must raise e to the power 0 to get 1.
2. $\ln e = 1$	We must raise e to the power 1 to get e.
3. $\ln e^x = x$	We must raise e to the power x to get e^x.
4. $e^{\ln x} = x$	$\ln x$ is the power to which e must be raised to get x.

More examples..?

Examples

$\log_x 1000 = 3$

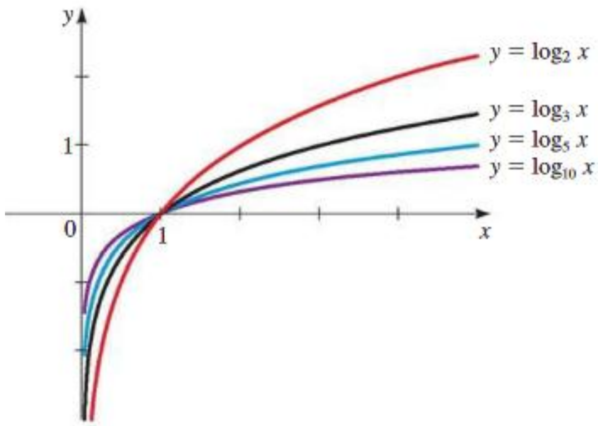
$\log \frac{1}{\sqrt[3]{10}} =$

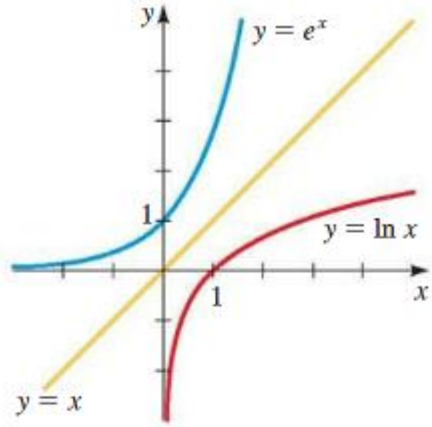
$\ln e^{10} =$

$\ln \frac{1}{e} =$

$e^{\ln \pi} =$

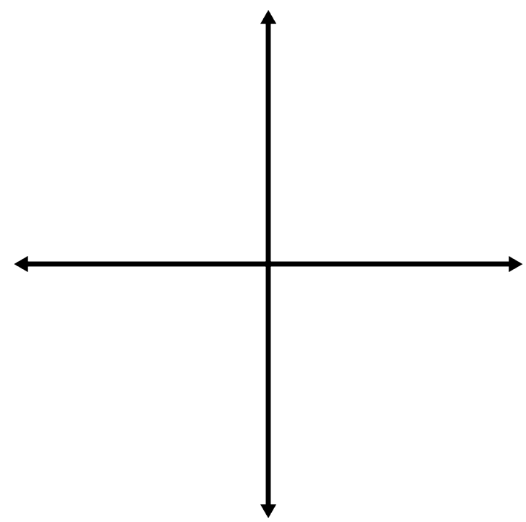
Graphing Logarithms





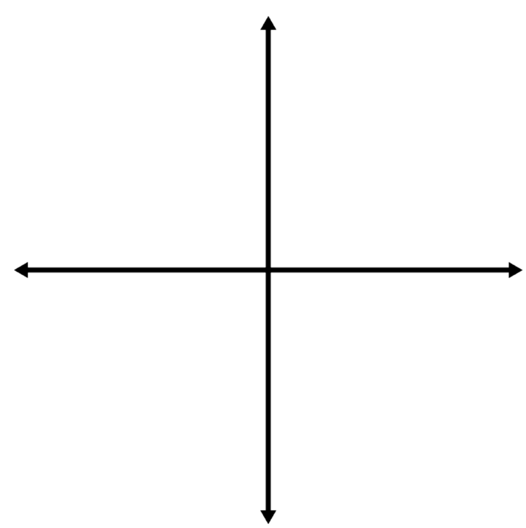
You can graph logarithms the same ways we have graphed everything else: using either a table or a transformation. In either, first transform the logarithm into its exponential form and then use the information found in the previous section.

Ex: Graph $y = 3^x$ and use it to graph $y = \log_3 x$



Ex: Graph the following function by transforming the graph found above

$$\boxed{y = \log_3(x-1) - 2}$$



4.4- Laws of Logarithms

Due to the fact that logarithms are equivalent to exponents, the following properties arise

LAWS OF LOGARITHMS

Let a be a positive number, with $a \neq 1$. Let A, B, and C be any real numbers with $A > 0$ and $B > 0$.

Law	Description
1. $\log_a(AB) = \log_a A + \log_a B$	The logarithm of a product of numbers is the sum of the logarithms of the numbers.
2. $\log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B$	The logarithm of a quotient of numbers is the difference of the logarithms of the numbers.
3. $\log_a(A^C) = C\log_a A$	The logarithm of a power of a number is the exponent times the logarithm of the number.

$\log_2 2x =$

$\ln(\frac{7}{3}) =$

$\log 12^4 =$

When approaching logarithm problems, keep in mind the forms shown above. If the problem you are working on is not given in one of the above forms you will need to manipulate the expression to get it into the desired form before using one of the laws above.

$\log_5 \frac{x}{\sqrt{5}}$	$\ln\sqrt{z} + \ln z^2$
$\log_4 6 + 5\log_4 5$	$\log_5(x^2 - 1) - \log_5(x - 1)$
$\log 12 + \frac{1}{2}\log 7 - \log 2$	$\log(\frac{a^2}{b^4\sqrt{c}})$
$\log_2 160 - \log_2 5$	$\log_6 260 - \log_6 3 - \log_6 2$

4.5- Exponential and Logarithmic Equations

Exponential Equations

We can use the properties of logarithms found in the previous section to solve exponential equations

GUIDELINES FOR SOLVING EXPONENTIAL EQUATIONS

1. Isolate the exponential expression on one side of the equation.
2. Take the logarithm of each side, then use the Laws of Logarithms to "bring down the exponent."
3. Solve for the variable.

$1 + 10^{-x} = 5$	$3^{2x-1} = 5$
$4(1 + 10^{5x}) = 9$	$e^{3-5x} = 16$
$7^{\frac{x}{2}} = 5^{1-x}$	$\dfrac{10}{1 + e^{-x}} = 2$

Sometimes the exponential equation may multiply exponential terms. In these cases you should either:

a) Re-write the equation as a quadratic and solve
   Or
b) Factor out any common factors and solve

Ex:

$e^{2x} - e^x - 6 = 0$	$x^2 10^x - x10^x = 2(10^x)$
$e^{2x} - 25 = 0$	$x^2 e^x + xe^x - e^x = 0$

Logarithmic Equations

We can use a similar reasoning to solve logarithmic equations

GUIDELINES FOR SOLVING LOGARITHMIC EQUATIONS

1. Isolate the logarithmic term on one side of the equation; you might first need to combine the logarithmic terms.
2. Write the equation in exponential form (or raise the base to each side of the equation).
3. Solve for the variable.

$\ln(x-1) + \ln(x+2) = 1$	$\log_5 x + \log_5(x+1) = \log_5 20$

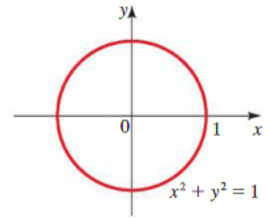
5.1- The Unit Circle

The Unit Circle

THE UNIT CIRCLE

The **unit circle** is the circle of radius 1 centered at the origin in the xy-plane. Its equation is

$$x^2 + y^2 = 1$$

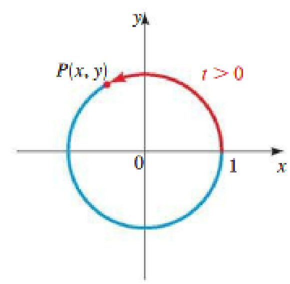


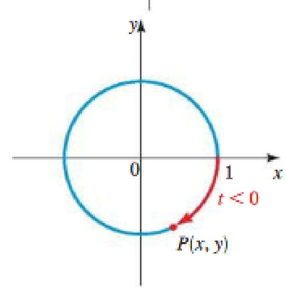
We know a point (x, y) is on the unit circle if $x^2 + y^2 = 1$

Given a single value known to be a coordinate on the unit circle, we can find the other coordinate by solving the equation shown above

The point $\left(x, -\frac{1}{3}\right)$ is on the unit circle. Find the x-coordinate given that it is positive	The x-coordinate of the point is $-\frac{2}{5}$ and the y-coordinate lies above the x-axis. Find the point.

Terminal Points on the Unit Circle





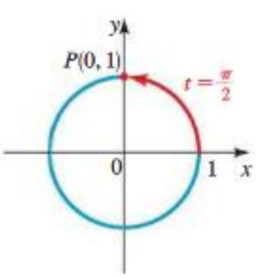
Let *t* be a distance on the unit circle.

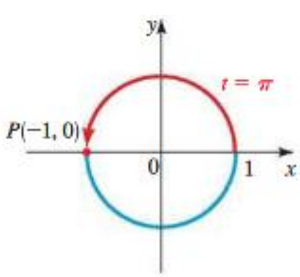
If <u>$t > 0$</u> then *t* represents a distance in the <u>counterclockwise</u> direction

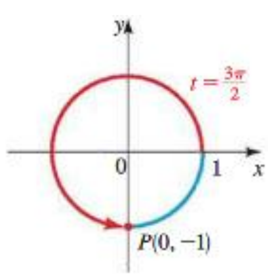
If <u>$t < 0$</u> then *t* represents a distance in the <u>clockwise</u> direction.

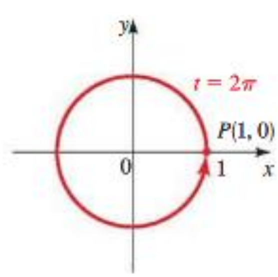
We call any point (x, y) obtained in this manner using a real number *t* a <u>terminal point</u>

Circumference= $2\pi r$, so circumference of the unit circle =





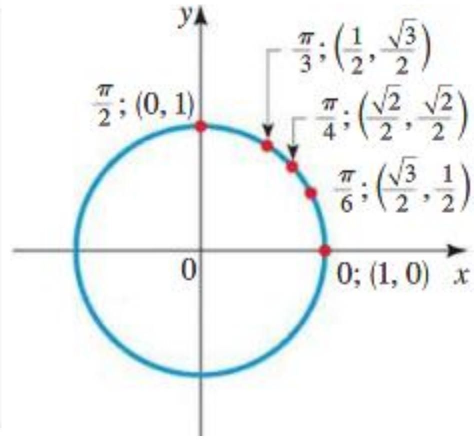




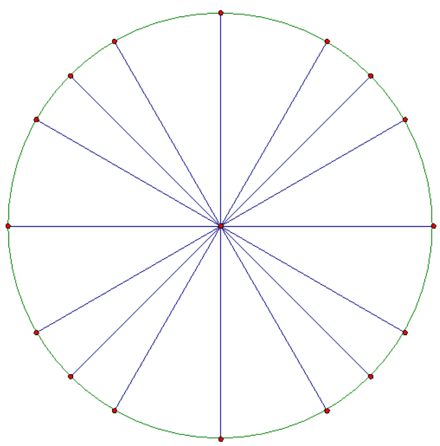
There are 5 distances, *t,* each associated with a terminal point that you must memorize (shown to the right).

We can use these 5 points to figure out the other 11 points around the circle (shown below)

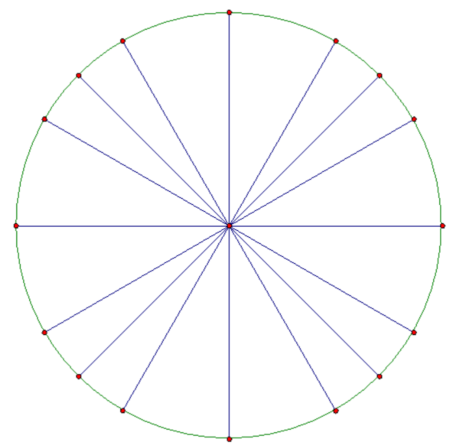
t	Terminal point determined by t
0	$(1,0)$
$\frac{\pi}{6}$	$\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$
$\frac{\pi}{4}$	$\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$
$\frac{\pi}{3}$	$\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$
$\frac{\pi}{2}$	$(0,1)$



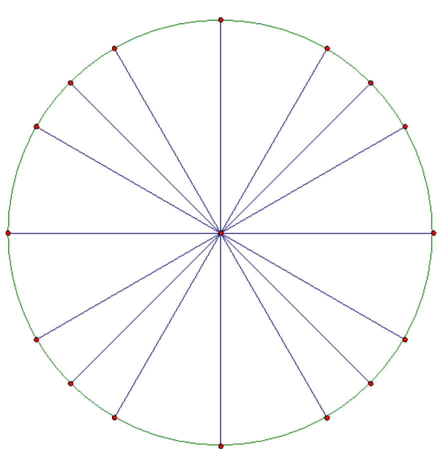
a) Increment: $\boldsymbol{\frac{\pi}{2}}$



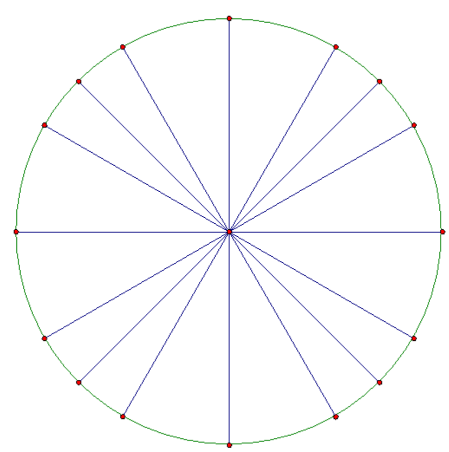
b) **Increment:** $\boldsymbol{\frac{\pi}{3}}$



c) Increment: $\boldsymbol{\frac{\pi}{4}}$



a) Increment: $\boldsymbol{\frac{\pi}{6}}$



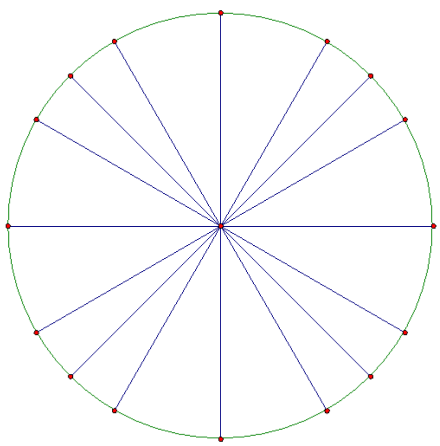
The Reference Number

REFERENCE NUMBER

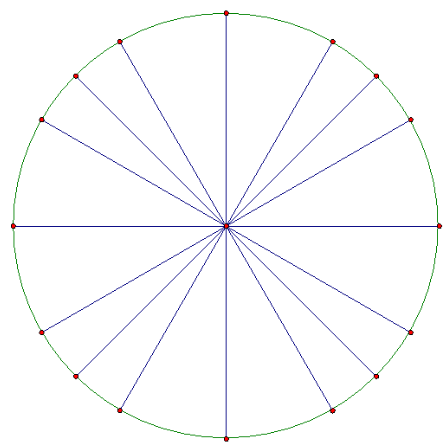
Let t be a real number. The **reference number** $\bar{t}$ associated with t is the shortest distance along the unit circle between the terminal point determined by t and the x-axis.

To find a reference number, trace out *t* along the unit circle and figure out the shortest distance between *t* and the x-axis.

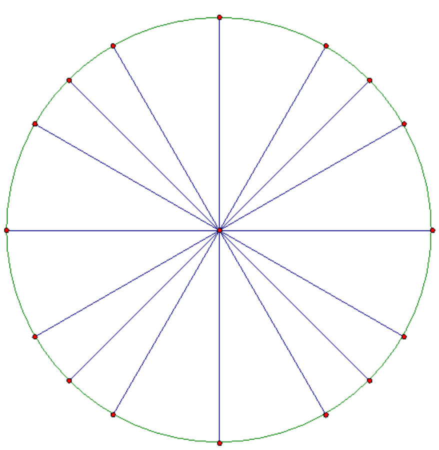
Find the reference number if $t = \frac{4\pi}{3}$



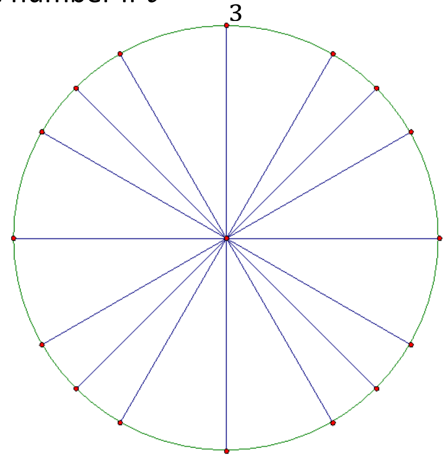
Find the reference number if $t = \frac{-7\pi}{6}$



Find the reference number if $t = \frac{17\pi}{4}$



Find the reference number if $t = \frac{-5\pi}{3}$



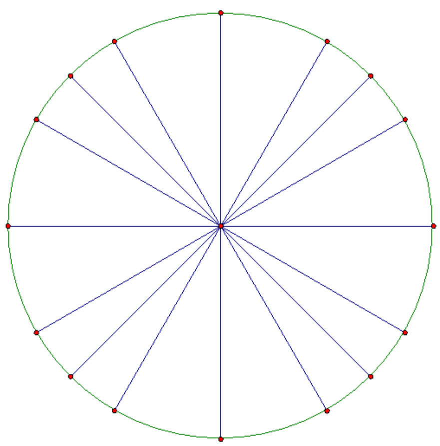
Combining each concept seen above, we can find the terminal point given any *t*

USING REFERENCE NUMBERS TO FIND TERMINAL POINTS

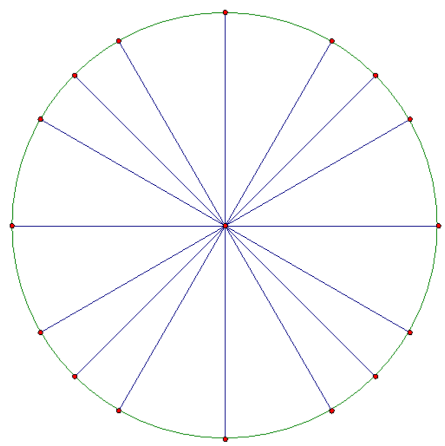
To find the terminal point P determined by any value of t, we use the following steps:

1. Find the reference number $\bar{t}$.
2. Find the terminal point $Q(a, b)$ determined by $\bar{t}$.
3. The terminal point determined by t is $P(\pm a, \pm b)$, where the signs are chosen according to the quadrant in which this terminal point lies.

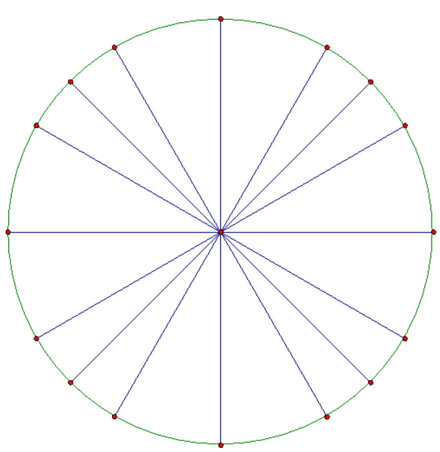
Find the terminal point if $t = \frac{7\pi}{6}$



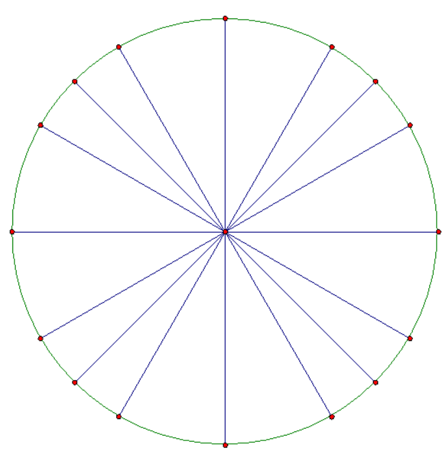
Find the terminal point if $t = \frac{-7\pi}{4}$



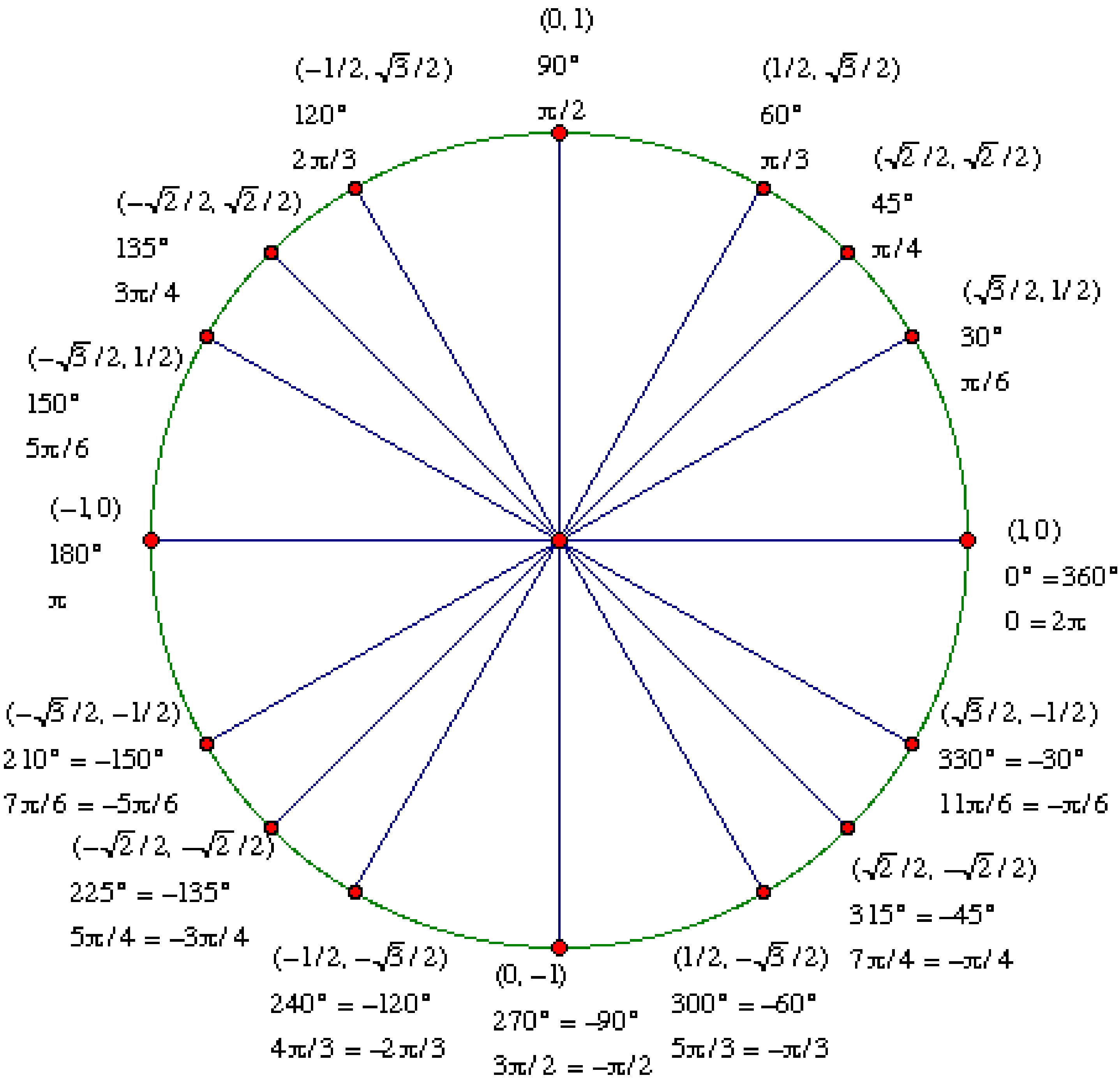
Find the terminal point if $t = \frac{2\pi}{3}$



Find the terminal point if $t = \frac{-13\pi}{6}$



Here is a completed unit circle that combines all the information from the second page. Use this picture only as a reference to check your reasoning. That is, you need to be able to find and label any of these points without referencing this picture, so only use this picture to check your reasoning.



5.2- Trigonometric Functions of Real Numbers

For the remainder of chapter 5, the unit circle concepts found in the previous section (5.1) are used

The Trigonometric Functions

Periodic behavior (a behavior that repeats itself over and over again) is common in nature. The rising and setting of the sun, the flux of the tides, animal populations etc. are all examples of periodic behavior. In order to describe these types of behavior we need functions that are periodic.

DEFINITION OF THE TRIGONOMETRIC FUNCTIONS

Let t be any real number and let $P(x, y)$ be the terminal point on the unit circle determined by t. We define

$$\sin t = y \qquad \cos t = x \qquad \tan t = \frac{y}{x} \quad (x \neq 0)$$

$$\csc t = \frac{1}{y} \quad (y \neq 0) \qquad \sec t = \frac{1}{x} \quad (x \neq 0) \qquad \cot t = \frac{x}{y} \quad (y \neq 0)$$

Given a number *t*, the values of the above trig functions are based on the terminal point (x,y) on the unit circle that is associated with *t*

Ex: Find the exact function values for all six trigonometric functions given the real number $t = -\frac{\pi}{3}$

First find the terminal point:

Ex: Find the exact function value for all six trigonometric functions given the real number $t = \frac{7\pi}{6}$

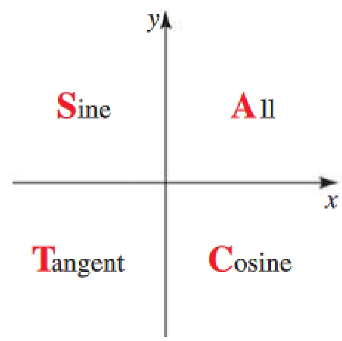
A complete list of trig functions and values for the points we will use is located on pg 378. However, do not get used to using this table, only use it as a reference to check your work.

DOMAINS OF THE TRIGONOMETRIC FUNCTIONS

Function	Domain
sin, cos	All real numbers
tan, sec	All real numbers other than $\frac{\pi}{2} + n\pi$ for any integer n
cot, csc	All real numbers other than $n\pi$ for any integer, n

SIGNS OF THE TRIGONOMETRIC FUNCTIONS

Quadrant	Positive Functions	Negative Functions
I	all	none
II	sin, csc	cos, sec, tan, cot
III	tan, cot	sin, csc, cos, sec
IV	cos, sec	sin, csc, tan, cot



Reasoning Behind Domains and Signs:

EVEN-ODD PROPERTIES

Sine, cosecant, tangent, and cotangent are odd functions; cosine and secant are even functions.

$$\sin(-t) = -\sin t \qquad \cos(-t) = \cos t \qquad \tan(-t) = -\tan t$$

$$\csc(-t) = -\csc t \qquad \sec(-t) = \sec t \qquad \cot(-t) = -\cot t$$

Ex: $\sin\left(-\frac{\pi}{3}\right) = -\sin(\frac{\pi}{3})$, $\cos\left(-\frac{2\pi}{3}\right) = \cos(\frac{2\pi}{3})$, $\csc\left(\frac{\pi}{6}\right) = -\csc(-\frac{\pi}{6})$

FUNDAMENTAL IDENTITIES

Reciprocal Identities

$$\csc t = \frac{1}{\sin t} \qquad \sec t = \frac{1}{\cos t} \qquad \cot t = \frac{1}{\tan t} \qquad \tan t = \frac{\sin t}{\cos t} \qquad \cot t = \frac{\cos t}{\sin t}$$

Pythagorean Identities

$$\sin^2 t + \cos^2 t = 1 \qquad \tan^2 t + 1 = \sec^2 t \qquad 1 + \cot^2 t = \csc^2 t$$

We can use all of the information on the previous pages to answer the following questions:

1) Find the sign of the expression if the terminal point determined by *t* is in the given quadrant

a) $\tan t \sec t$ in Quadrant IV

b) $\cos t \sec t$ in any Quadrant

2) From the information given, find the quadrant in which the terminal point determined by *t* lies

a) $\tan t > 0$ and $\sin t < 0$

b) $\cos t < 0$ and $\cot t < 0$

More practice..

3) Write the first expression in terms of the second if the terminal point determined by *t* is in the given quadrant

a) $\cos t$, $\sin t$ in Quadrant IV

b) $\sin t$, $\sec t$ in Quadrant IV

4) Find the values of all six trigonometric functions of *t* from the given information

a) $\cos t = -\frac{4}{5}$, terminal point of t is in Quadrant III

b) $\sec t = 2$, $\sin t < 0$

<u>5.3- Trigonometric Graphs (Sine and Cosine)</u>

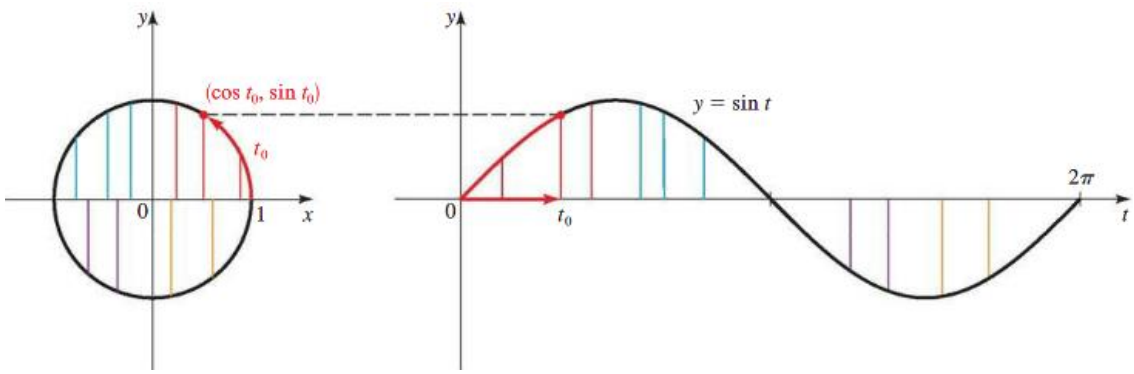
<u>This section focuses on graphs of **sine** and **cosine** functions. Others are in the next section</u>

PERIODIC PROPERTIES OF SINE AND COSINE

The functions sine and cosine have period 2π:

$$\sin(t + 2\pi) = \sin t \qquad\qquad \cos(t + 2\pi) = \cos t$$

Meaning:



<u>General forms of Sine and Cosine</u>

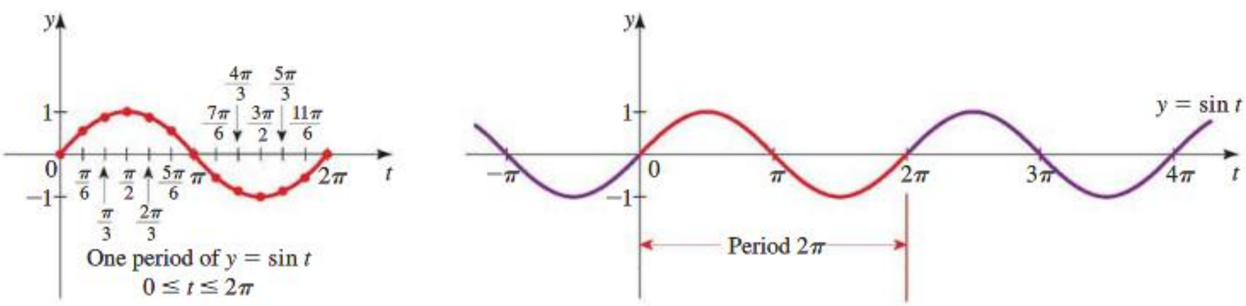


FIGURE 2 Graph of sin t

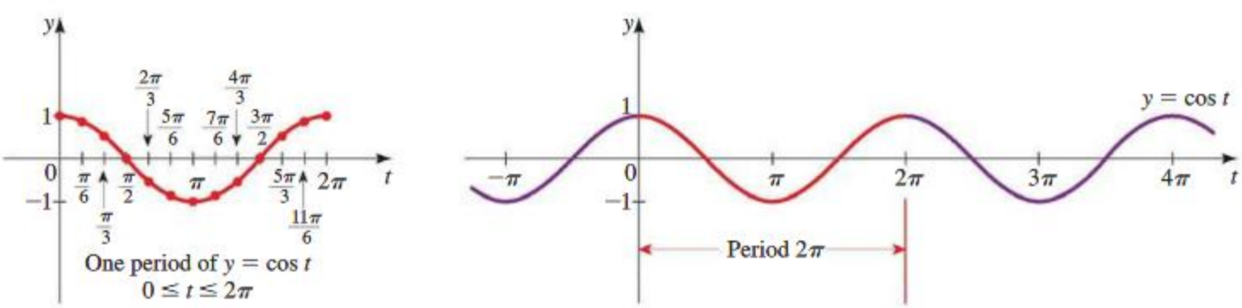


FIGURE 3 Graph of cos t

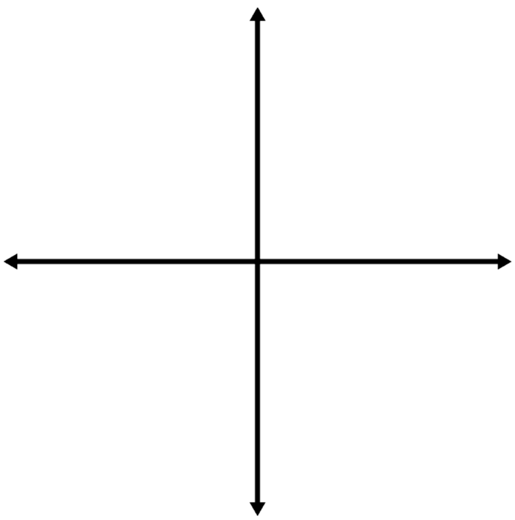
<u>Graphs of Transformations of Sine and Cosine</u>

We can use the techniques found in section 2.5 paired with the general forms of Sine and Cosine seen above to draw transformations of the graphs sin *t* and cos *t*.

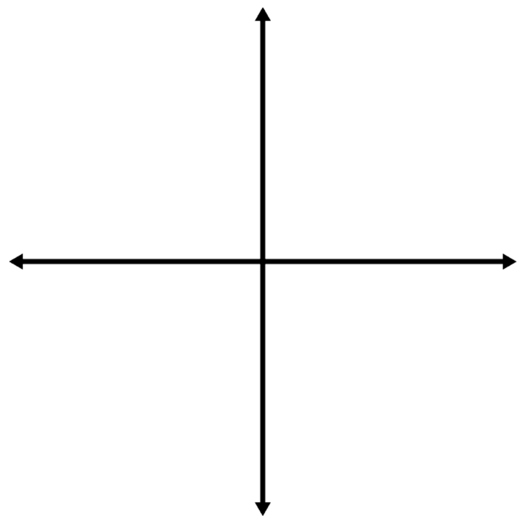
<u>Form 1:</u> $\boldsymbol{y = c + a \sin x}$ **or** $\boldsymbol{y = c + a \cos x}$

<u>Explanation of form:</u>

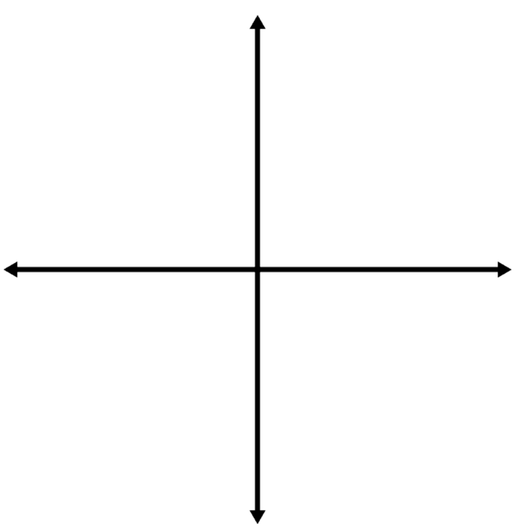
$y = 4 + \sin x$



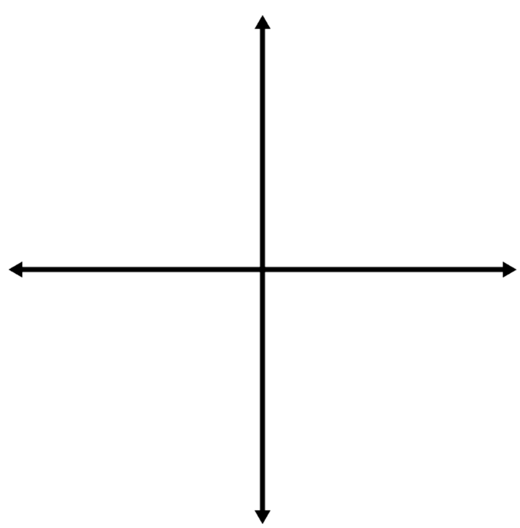
$y = 1 - \cos x$



$y = 2 \sin x$



$y = 2 - \frac{1}{2} \cos x$



Form 2: $\boldsymbol{y = c + a \sin kx}$ **or** $\boldsymbol{y = c + a \cos kx}$

SINE AND COSINE CURVES

The sine and cosine curves

$$y = a \sin kx \qquad \text{and} \qquad y = a \cos kx \qquad (k > 0)$$

have **amplitude** $|a|$ and **period** $2\pi/k$.

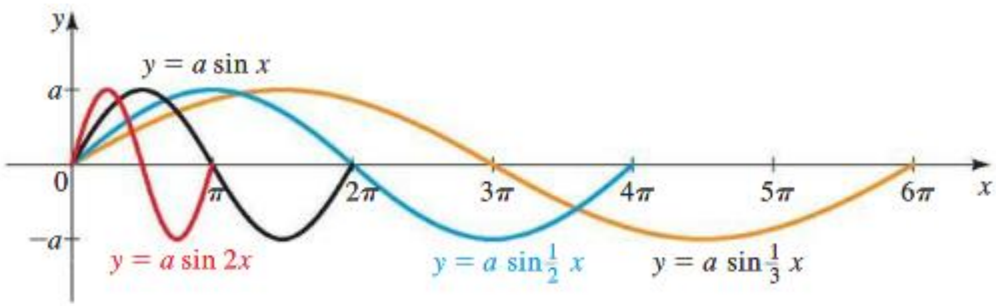
An appropriate interval on which to graph one complete period is $[0, 2\pi/k]$.

Amplitude- the height of the graph

Period- How long it takes for the function to cycle through its y-values. That is, how long until the function starts to repeat its y-values

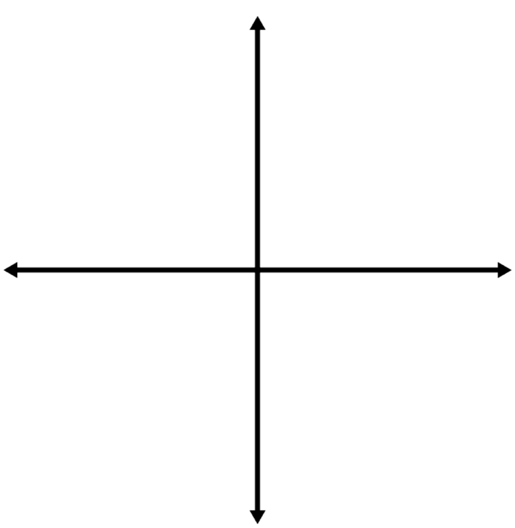
Explanation of form:

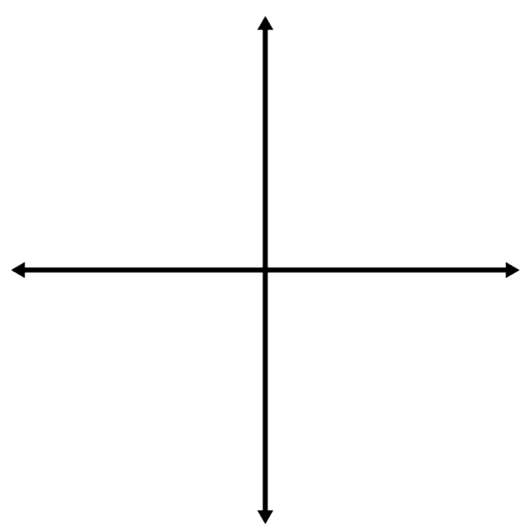
Visual Example using $y = \text{a} \sin x$:



$y = \frac{1}{2} \cos 4x$

$y = 1 - 3 \sin \pi x$





Form 3: $y = c + a \sin k(x - b)$ **or** $y = c + a \cos k(x - b)$

SHIFTED SINE AND COSINE CURVES

The sine and cosine curves

$$y = a \sin k(x - b) \quad \text{and} \quad y = a \cos k(x - b) \quad (k > 0)$$

have **amplitude** $|a|$, **period** $2\pi/k$, and **phase shift** b.

An appropriate interval on which to graph one complete period is $[b, b + (2\pi/k)]$.

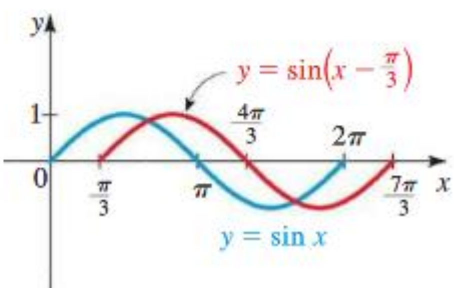
Amplitude- largest value that the function obtains

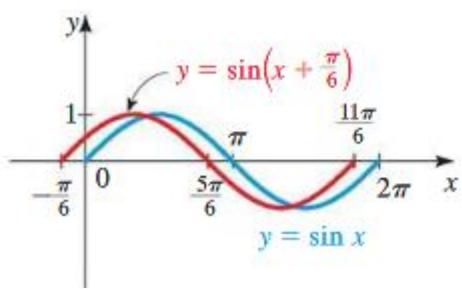
Period- How long it takes for the function to cycle through its y-values. That is, how long until the function starts to repeat its y-values

Phase Shift- a horizontal shift in the graph *b* units

Explanation of form:

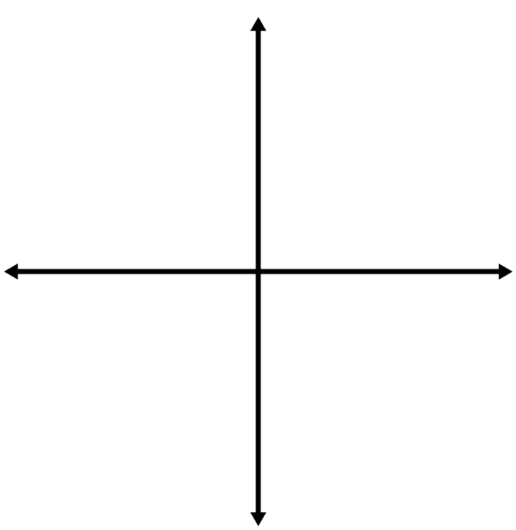
Visual Reference:

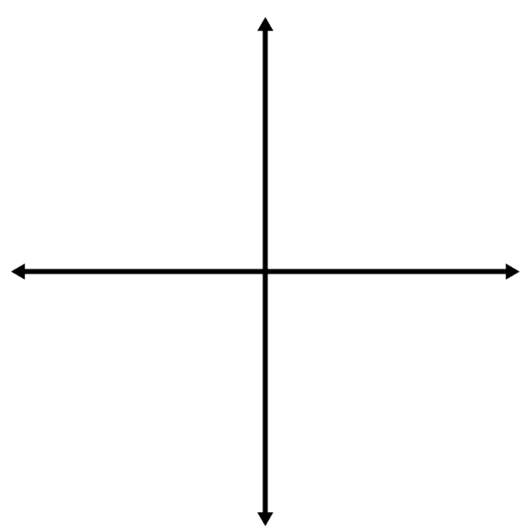




$y = 2\sin(x - \frac{\pi}{3})$

$y = 1 + \cos(3x + \frac{\pi}{2})$





5.4- More Trigonometric Graphs

Graphs of Tangent, Cotangent, Secant and Cosecant

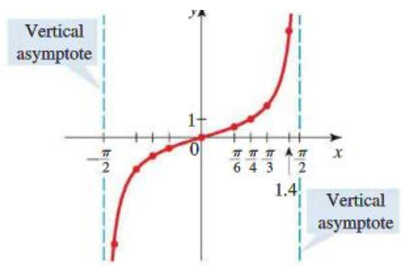


FIGURE 1
One period of $y = \tan x$

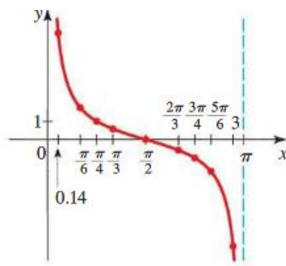


FIGURE 2
One period of $y = \cot x$

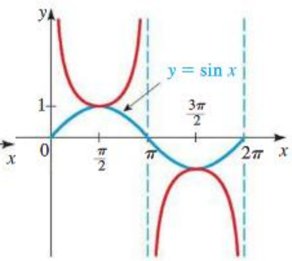


FIGURE 3
One period of $y = \csc x$

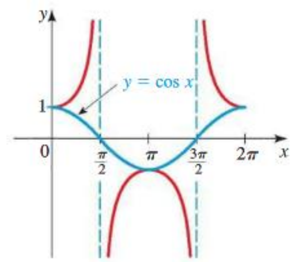


FIGURE 4
One period of $y = \sec x$

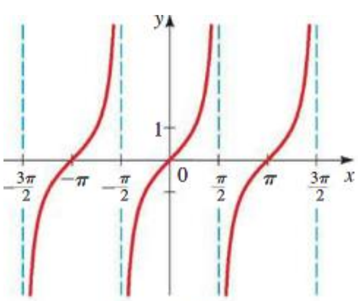
PERIODIC PROPERTIES

The functions tangent and cotangent have period π:

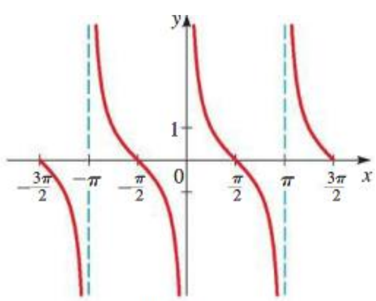
$$\tan(x + \pi) = \tan x \qquad \cot(x + \pi) = \cot x$$

The functions cosecant and secant have period 2π:

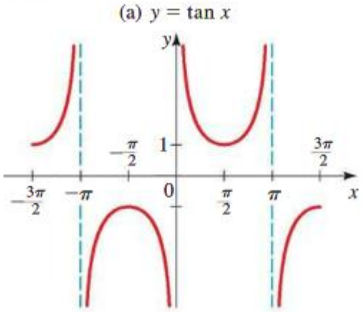
$$\csc(x + 2\pi) = \csc x \qquad \sec(x + 2\pi) = \sec x$$



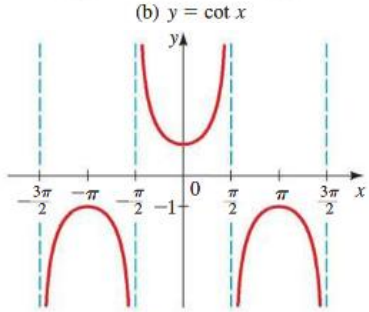
(a) $y = \tan x$



(b) $y = \cot x$



(c) $y = \csc x$



(d) $y = \sec x$

Explanation:

Transformations of Tangent and Cotangent

TANGENT AND COTANGENT CURVES

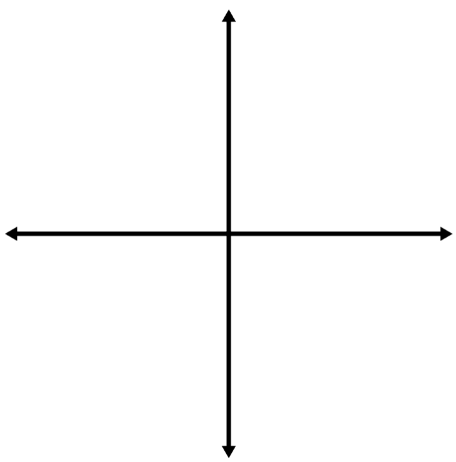
The functions

$$y = a \tan kx \qquad \text{and} \qquad y = a \cot kx \qquad (k > 0)$$

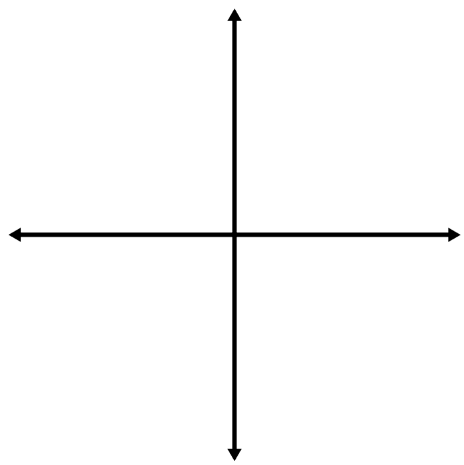
have period π/k.

Using information from section 2.5 paired with the general forms seen above we can graph these functions

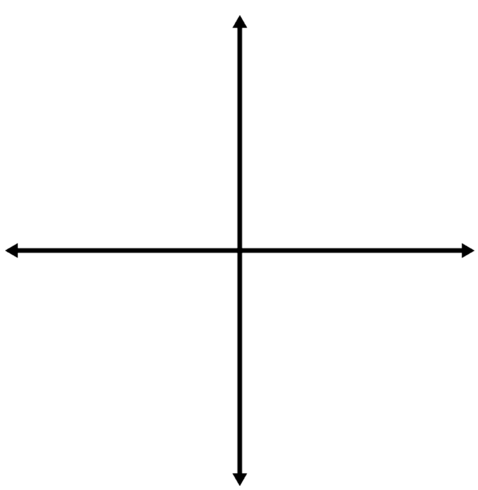
$y = -4 \tan x$



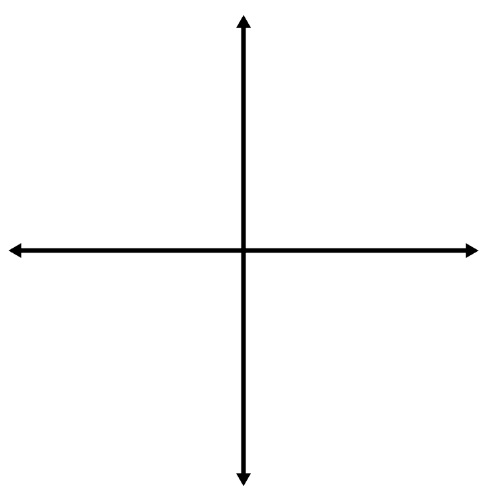
$y = \frac{1}{2} \cot x$



$y = \frac{1}{2} \tan\left(x - \frac{\pi}{4}\right)$



$y = \cot(\frac{\pi}{2} x)$



Transformations of Cosecant and Secant

COSECANT AND SECANT CURVES

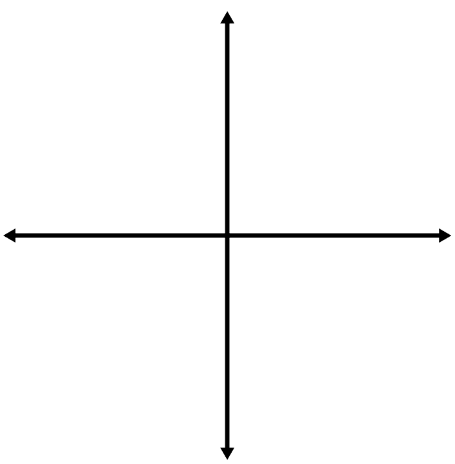
The functions

$$y = a \csc kx \qquad \text{and} \qquad y = a \sec kx \qquad (k > 0)$$

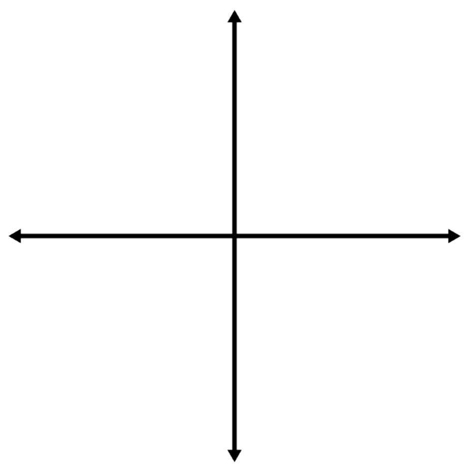
have period $2\pi/k$.

Using information from section 2.5 paired with the general forms seen on the first page we can graph these functions

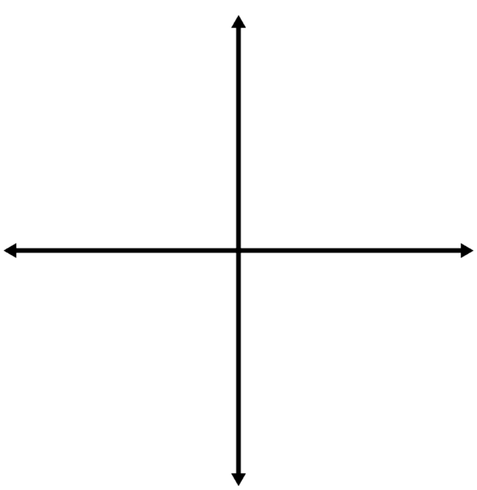
$y = \frac{1}{2}\csc x$



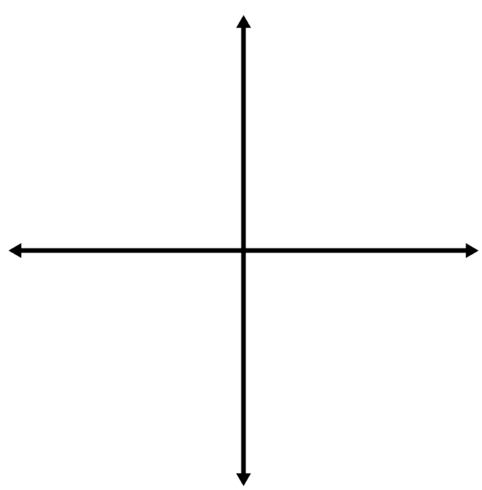
$y = -3\sec x$



$y = 2\csc(x - \frac{\pi}{3})$



$y = 5\sec(2\pi x)$



5.5- Inverse Trig Functions and Their Graphs

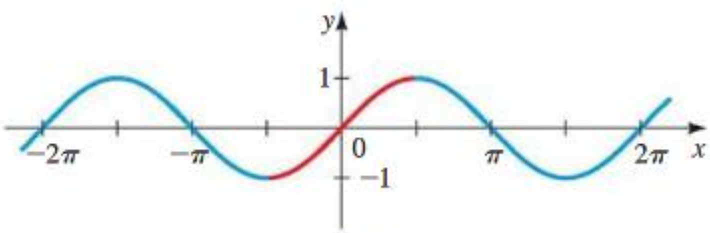
Recall, an inverse function of a graph sends all of its y-values back to the x-values from which they came.

That is, if $f(x) = y$, then $f^{-1}(y) = x$.

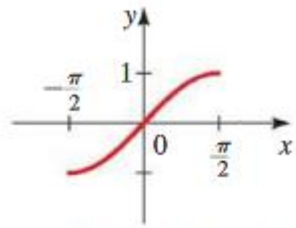
However, in order for this to work the function needs to be one-to-one.

Although Trig functions are not one-to-one, if we restrict their domain we can make them one-to-one on certain intervals, which will result in each having an inverse on a given interval.

Arcsine: The Inverse Sine Function



$y = \sin x$



$y = \sin x,\ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

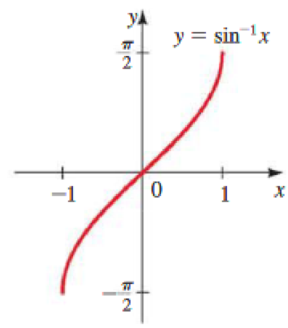


FIGURE 2 Graph of $y = \sin^{-1} x$

DEFINITION OF THE INVERSE SINE FUNCTION

The **inverse sine function** is the function $\sin^{-1}$ with domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$ defined by

$$\boxed{\sin t = y \quad \Leftrightarrow \quad \sin^{-1} y = t}$$

The inverse sine function is also called **arcsine**, denoted by **arcsin**.

$$\sin(\sin^{-1} x) = x \quad \text{for} \quad -1 \le x \le 1$$

$$\sin^{-1}(\sin x) = x \quad \text{for} \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

$\sin^{-1}(-1)$

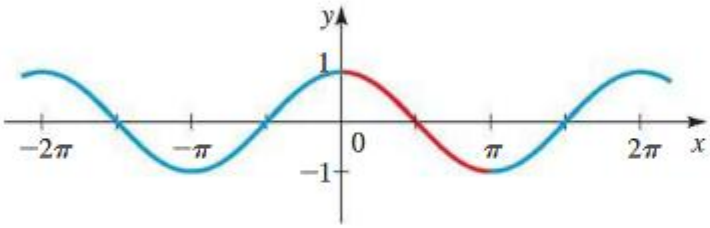
$\sin^{-1}(\sin(-\frac{\pi}{4}))$

$\sin(\sin^{-1}(5))$

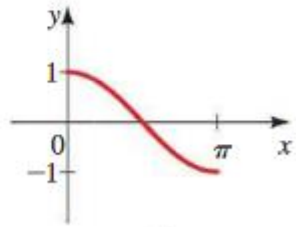
$\sin^{-1}(\frac{\sqrt{2}}{2})$

$\sin^{-1}(\sin(\frac{11\pi}{4}))$

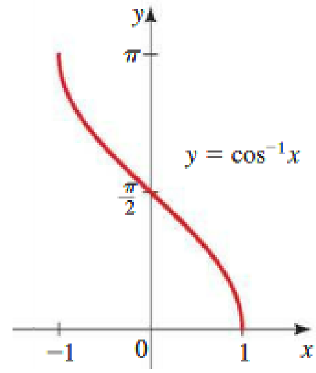
Arccosine: The Inverse Cosine Function



$y = \cos x$



$y = \cos x,\ 0 \le x \le \pi$



DEFINITION OF THE INVERSE COSINE FUNCTION

The **inverse cosine function** is the function $\cos^{-1}$ with domain $[-1, 1]$ and range $[0, \pi]$ defined by

$$\boxed{\cos t = x \quad \Leftrightarrow \quad \cos^{-1} x = t}$$

The inverse cosine function is also called **arccosine**, denoted by **arccos**.

$\cos^{-1}(-1)$

$\cos^{-1}(-\frac{\sqrt{2}}{2})$

$\cos(\cos^{-1} x) = x \quad \text{for} \quad -1 \le x \le 1$

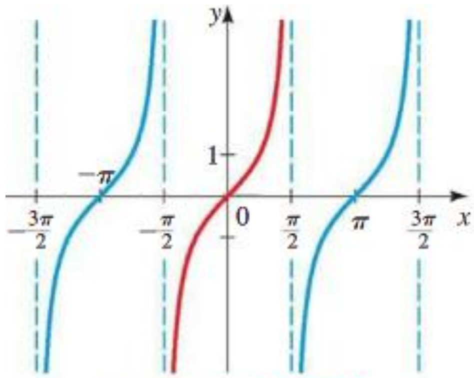
$\cos^{-1}(\cos x) = x \quad \text{for} \quad 0 \le x \le \pi$

$\cos^{-1}(\cos(-\frac{\pi}{6}))$

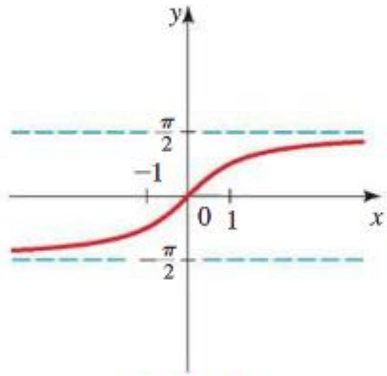
$\cos(\cos^{-1}(-\frac{2}{3}))$

$\cos^{-1}(\cos(\frac{17\pi}{6}))$

Arctangent: The Inverse Tangent Function



$y = \tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$



$y = \tan^{-1} x$

DEFINITION OF THE INVERSE TANGENT FUNCTION The **inverse tangent function** is the function $\tan^{-1}$ with domain $\mathbb{R}$ and range $(-\pi/2, \pi/2)$ defined by $\boxed{\tan t = \frac{y}{x} \quad \Leftrightarrow \quad \tan^{-1}\frac{y}{x} = t}$ The inverse tangent function is also called **arctangent**, denoted by **arctan**.	$\tan(\tan^{-1} x) = x \quad \text{for} \quad x \in \mathbb{R}$ $\tan^{-1}(\tan x) = x \quad \text{for} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$
	$\tan^{-1}(\tan(-\frac{\pi}{4}))$
$\tan^{-1}(-\frac{1}{\sqrt{3}})$	
	$\tan(\tan^{-1}(9846))$
$\tan^{-1}(-1)$	
	$\tan^{-1}(\tan(\frac{4\pi}{3})$
$\tan^{-1}(\sqrt{3})$	

6.1- Angle Measure

Angle Measure

DEFINITION OF RADIAN MEASURE

If a circle of radius 1 is drawn with the vertex of an angle at its center, then the measure of this angle in **radians** (abbreviated **rad**) is the length of the arc that subtends the angle (see Figure 2).

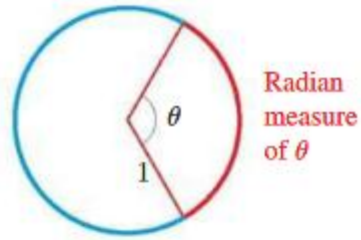
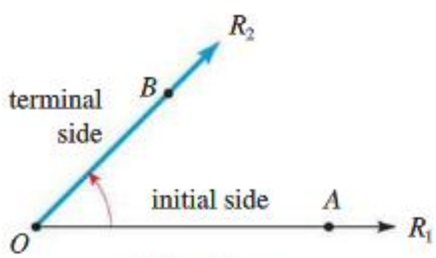
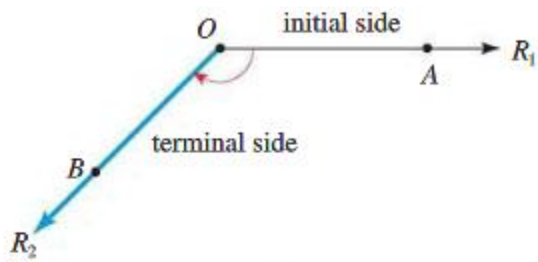


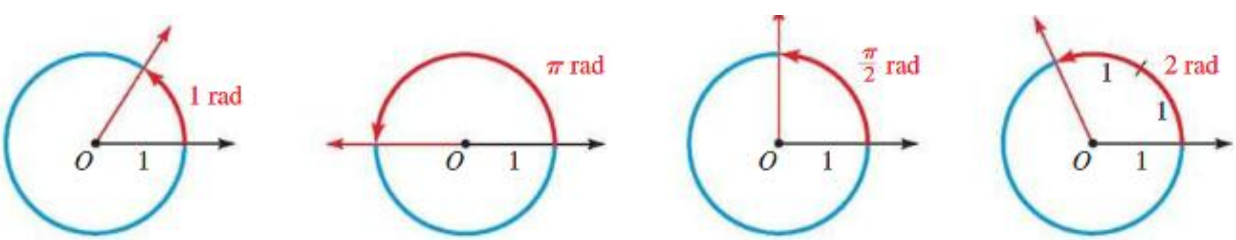
FIGURE 2



Positive angle



Negative angle



Since a complete revolution in degrees is 360° and in radians is 2π, the following relationship exists:

RELATIONSHIP BETWEEN DEGREES AND RADIANS

$$180^\circ = \pi \text{ rad} \qquad 1 \text{ rad} = \left(\frac{180}{\pi}\right)^\circ \qquad 1^\circ = \frac{\pi}{180} \text{ rad}$$

1. To convert degrees to radians, multiply by $\frac{\pi}{180}$.
2. To convert radians to degrees, multiply by $\frac{180}{\pi}$.

Convert the angles given in degrees to angles in radians:

54°

-300°

Convert the angles given in radians to angles in degrees:

$\frac{11\pi}{3}$ $\qquad$ $-\frac{3\pi}{2}$

Angles in Standard Position

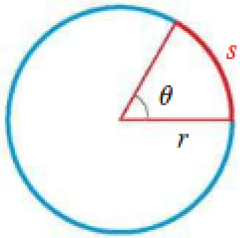
An angle is in <u>standard position</u> if it is drawn in the xy-plane with its vertex at the origin and its initial side on the positive x-axis

Two angles in standard position are <u>coterminal</u> if their sides coincide

- Given an angle in <u>radians</u> one can determine other positive/negative angles that are coterminal by adding/subtracting multiples of 2π from the given angle
- Given an angle in <u>degrees</u> one can determine other positive/negative angles that are coterminal by adding/subtracting multiples of 360° from the given angle

Let $\theta = \frac{11\pi}{6}$, find some positive and negative coterminal angles	Let $\theta = -45°$, find some positive and negative coterminal angles

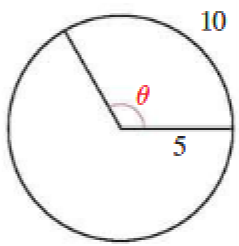
Length of a Circular Arc



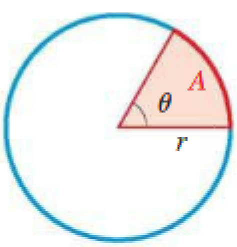
LENGTH OF A CIRCULAR ARC

In a circle of radius r, the length s of an arc that subtends a central angle of θ radians is

$$s = r\theta$$

Find the angle θ in the figure. 	Find the length of an arc that subtends a central angle of 45°in a circle of radius 10 m	A central angle θ in a circle of radius 5 m is subtended by an arc of length 6 m. Find the measure of θ in degrees and radians

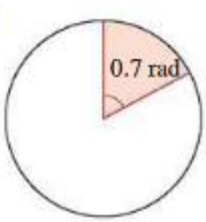
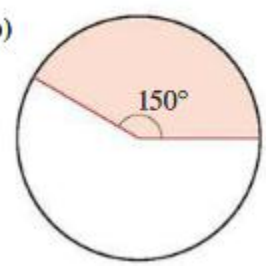
Area of a Circular Sector



AREA OF A CIRCULAR SECTOR

In a circle of radius r, the area A of a sector with a central angle of θ radians is

$$A = \frac{1}{2}r^2\theta$$

Find the radius of each circle if the area of the sector is 12. **(a)**  **(b)** 	If a circle has a sector of area 10 and radius 2 calculate the central angle θ in radians and degrees

6.2- Trigonometry of Right Triangles

Trigonometric Ratios

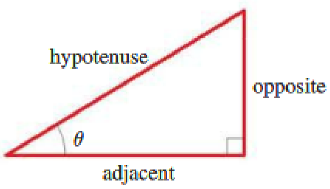


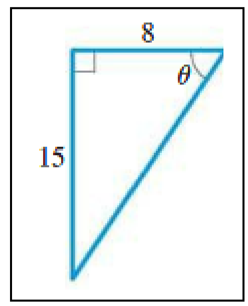
FIGURE 1

THE TRIGONOMETRIC RATIOS

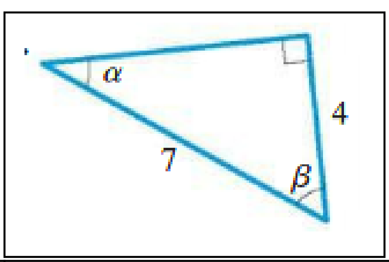
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

Find the exact values of the six trigonometric ratios of the angle θ in the triangle



Consider the following triangle:



b) Find $\sin\alpha$ and $\cos\beta$	a) Find $\tan\alpha$ and $\cot\beta$	c) Find $\sec\alpha$ and $\csc\beta$

Special Triangles

You should remember these special angle ratios because they are the ones that we will use:

θ in degrees	θ in radians	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

Application of Trigonometry of Right Triangles

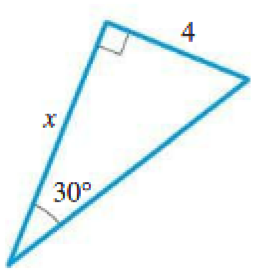
Given a trigonometric ratio we can construct the triangle

Ex: If $\cos\theta = \frac{9}{40}$, sketch a triangle that has acute angle θ

If a question asks you to "solve the right triangle" it means find all side lengths and all angles.

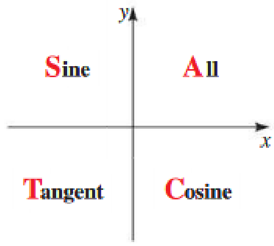
We can do this using all of the above properties paired with the Pythagorean Theorem

Ex: Solve the following right triangle



6.3- Trigonometric Functions of Angles

A few reminders:



You can remember this as "All Students Take Calculus."

SIGNS OF THE TRIGONOMETRIC FUNCTIONS

Quadrant	Positive Functions	Negative Functions
I	all	none
II	sin, csc	cos, sec, tan, cot
III	tan, cot	sin, csc, cos, sec
IV	cos, sec	sin, csc, tan, cot

REFERENCE ANGLE

Let θ be an angle in standard position. The **reference angle** $\overline{\theta}$ associated with θ is the acute angle formed by the terminal side of θ and the x-axis.

To convert degrees to radians, multiply by $\frac{\pi}{180}$

We can use the above techniques to determine the value of any trig function given an angle θ

EVALUATING TRIGONOMETRIC FUNCTIONS FOR ANY ANGLE

To find the values of the trigonometric functions for any angle θ, we carry out the following steps.

1. Find the reference angle $\overline{\theta}$ associated with the angle θ.
2. Determine the sign of the trigonometric function of θ by noting the quadrant in which θ lies.
3. The value of the trigonometric function of θ is the same, except possibly for sign, as the value of the trigonometric function of $\overline{\theta}$.

Ex-Find the exact values of the following functions:

$\sin 225°$

$\tan \frac{5\pi}{6}$

FUNDAMENTAL IDENTITIES

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

Using these fundamental identities, we can make substitutions to write one trig function in terms of another.

Write $\cot\theta$ in terms of $\sin\theta$ given that θ is in Quadrant II

Write $\sec\theta$ in terms of $\sin\theta$ given that θ is in Quadrant I

7.1- Trigonometric Identities

Many of the following identities we studied in previous sections:

FUNDAMENTAL TRIGONOMETRIC IDENTITIES

Reciprocal Identities

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$

Even-Odd Identities

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x \qquad \tan(-x) = -\tan x$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - u\right) = \cos u \qquad \tan\left(\frac{\pi}{2} - u\right) = \cot u \qquad \sec\left(\frac{\pi}{2} - u\right) = \csc u$$

$$\cos\left(\frac{\pi}{2} - u\right) = \sin u \qquad \cot\left(\frac{\pi}{2} - u\right) = \tan u \qquad \csc\left(\frac{\pi}{2} - u\right) = \sec u$$

In the following examples, rewrite the expression in terms of sine and cosine and then simplify the expression.

Work	Substitution
$\tan(\theta)\csc(\theta)$	
$\dfrac{\sec(x) - \cos(x)}{\tan(x)}$	

Proving Trigonometric Identities

Many identities arrise from the fundamental identities shown on the first page. The idea behind proving an identity is that we are given two things that are said to be equal and we want to start with one side of the identity and transform it into the other side of the identity using substitutions from page 1.

GUIDELINES FOR PROVING TRIGONOMETRIC IDENTITIES

1. **Start with one side.** Pick one side of the equation and write it down. Your goal is to transform it into the other side. It's usually easier to start with the more complicated side.
2. **Use known identities.** Use algebra and the identities you know to change the side you started with. Bring fractional expressions to a common denominator, factor, and use the fundamental identities to simplify expressions.
3. **Convert to sines and cosines.** If you are stuck, you may find it helpful to rewrite all functions in terms of sines and cosines.

If you get stuck along the way try working backwards. That is, try starting from the less complicated side and work back towards the more complicated side.

Work	Reason
Show $\cos(-x) - \sin(-x) = \cos(x) + \sin(x)$	

Work	Reason
Show $\frac{\cos(x)}{\sec(x)} + \frac{\sin(x)}{\csc(x)} = 1$	

Work	Reason
Show $\frac{1+\tan^2 x}{1-\tan^2 x} = \frac{1}{\cos^2 x - \sin^2 x}$	

Work	Reason
Show $(\cot x - \csc x)(\cos x + 1) = -\sin x$	

Notice the following "hidden identities":

- If given $(1 - \sin x)$, notice that $(1 - \sin x)(1 + \sin x) = 1 - \sin^2 x$ and since $\sin^2 x + \cos^2 x = 1$, $1 - \sin^2 x = \cos^2 x$ and thus $(1 - \sin x)(1 + \sin x) = 1 - \sin^2 x = \cos^2 x$

- Similarly, if given $(1 - \cos x)$ then $(1 - \cos x)(1 + \cos x) = 1 - \cos^2 x = \sin^2 x$

- Also, if given $(\sec x - 1)$, then $(\sec x - 1)(\sec x + 1) = \sec^2 x - 1 = \tan^2 x$. (since $\tan^2 x + 1 = \sec^2 x$)

- Similarly, if given $\csc x - 1$ then $(\csc x - 1)(\csc x + 1) = (\csc^2 x - 1) = \cot^2 x$

Moral: You can re-arrange the identities on the first page to create new identities.

Also: given $[1 \pm (\cos x / \sin x / \sec x / \csc x / others)]$ you may have to multiply by the conjugate to move forward in proving the identity.

Work	**Reason**
Show $\frac{1}{1-\sin x} - \frac{1}{1+\sin x} = 2 \sec x \tan x$	

Work	Reason
Show $\frac{\cos\theta}{1-\sin\theta} = \sec\theta + \tan\theta$	

Make the indicated substitution and simplify.

$\sqrt{1+x^2}$, $x = \tan\theta$. Assume that $0 \le \theta < \frac{\pi}{2}$

$\frac{1}{x^2\sqrt{4+x^2}}$, $x = 2\tan\theta$. Assume that $0 \le \theta < \frac{\pi}{2}$.

<u>7.2- Addition and Subtraction Formulas</u>

The following formulas <u>do not need to be memorized</u> as they will be provided on quizzes/tests/exams

ADDITION AND SUBTRACTION FORMULAS

Formulas for sine:
$$\sin(s + t) = \sin s \cos t + \cos s \sin t$$
$$\sin(s - t) = \sin s \cos t - \cos s \sin t$$

Formulas for cosine:
$$\cos(s + t) = \cos s \cos t - \sin s \sin t$$
$$\cos(s - t) = \cos s \cos t + \sin s \sin t$$

Formulas for tangent:
$$\tan(s + t) = \frac{\tan s + \tan t}{1 - \tan s \tan t}$$
$$\tan(s - t) = \frac{\tan s - \tan t}{1 + \tan s \tan t}$$

We can use the above identities to evaluate an unknown trig identity by transforming them to a more familiar form :

Evaluate $\cos(105°)$

Evaluate $\cos(10°)\cos(80°) - \sin(10°)\sin(80°)$

We can use the addition and subtraction formulas to prove identities.

Prove that $\cot\left(\frac{\pi}{2} - u\right) = \tan u$

Prove that $\cos(x + y) + \cos(x - y) = 2\cos x \cos y$

Prove that $\frac{\sin(x+y)-\sin(x-y)}{\cos(x+y)+\cos(x-y)} = \tan y$

7.3- Double-Angle, Half-Angle and Product-Sum Formulas

You **will** need to memorize the following formulas as they will **not** be provided on tests/exams

DOUBLE-ANGLE FORMULAS

Formula for sine: $\sin 2x = 2 \sin x \cos x$

Formulas for cosine:
$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= 1 - 2\sin^2 x \\ &= 2\cos^2 x - 1 \end{aligned}$$

Formula for tangent: $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$

We can use the Double-Angle Formulas to solve problems like this:

Ex:

Find $\sin(2x)$, $\cos(2x)$, and $\tan(2x)$ from the given information:

$\tan(x) = -\frac{4}{3}$ and x is in Quadrant II

FORMULAS FOR LOWERING POWERS

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

We can use the formulas for lowering powers to rewrite expression that involves high powers to those involving the first power of cosine.

Ex:

Use the formulas for lowering powers to rewrite the expression in terms of the first power of cosine.

$\cos^4 x$

HALF-ANGLE FORMULAS

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1 - \cos u}{2}} \qquad \cos\frac{u}{2} = \pm\sqrt{\frac{1 + \cos u}{2}}$$

$$\tan\frac{u}{2} = \frac{1 - \cos u}{\sin u} = \frac{\sin u}{1 + \cos u}$$

The choice of the + or − sign depends on the quadrant in which $u/2$ lies.

Use an appropriate Half-Angle formula to find the exact value of the expressions:

$\tan 15°$ $\sin 22.5°$

Prove the following Identities:

$$\sin(8x) = 2\sin(4x)\cos(4x)$$

$$\frac{1+\sin(2x)}{\sin(2x)} = 1 + \frac{1}{2}\sec(x)\csc(x)$$

<u>7.4 and 7.5- Trigonometric Equations</u>

7.4 and 7.5 can difficult, yet fun sections. They can be difficult because they combine many different trig ideas that we have studied this semester. They are fun because this is your opportunity to use your brain in creative ways!

<u>Think:</u> How could we solve the equation $\sin\theta = 1$?

<u>Note</u> In order for $\sin\theta = x$ and $\cos\theta = x$ to have solutions, we need $-1 \leq x \leq 1$. Otherwise, no solution exists

<u>Infinitely many solutions</u>

It is important to realize that if a trig equation has a single solution, then it has infinitely many solutions. Think about the unit circle and/or the graph of the trig functions.

In webassign/tests you will be asked for multiple values that are solutions. Keep in mind the unit circle and where appropriate values occur. For example…

Examples

$\tan\theta = 1$	Sometimes a trig function might be squared. $$4\sin^2\theta - 3 = 0$$
Sometimes you need to factor the equation $$4\cos^2\theta - 4\cos\theta + 1 = 0$$	Sometimes you need to factor the equation $$\sin^2\theta - \sin\theta - 2 = 0$$
Sometimes you need to solve for the θ inside the function $$\sin 3\theta = \frac{1}{2}$$	Sometimes you need to solve for the θ inside the function $$2\cos(2\theta) + 1$$

Sometimes you have to use one of the following double angle formulas or identities

DOUBLE-ANGLE FORMULAS

Formula for sine: $\sin 2x = 2 \sin x \cos x$

Formulas for cosine:
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 1 - 2\sin^2 x$$
$$= 2\cos^2 x - 1$$

Formula for tangent: $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$

$$\sin^2\theta + \cos^2\theta = 1 \quad \Leftrightarrow \quad \cos\theta = \sqrt{1 - \sin^2\theta} \quad \Leftrightarrow \quad \sin\theta = \sqrt{1 - \cos^2\theta}$$

$\cos 2\theta + \cos\theta = 2$	$\sin^2\theta = 4 - 2\cos^2\theta$
$2\sin^2\theta = 2 + \cos 2\theta$	